§26.2. Second Proof of Weyl's Character Formula

The title of this section is perhaps inaccurate: what we will give here is actually a sketch of the first proof of the Weyl character formula. Weyl, in his original proof, used what he called the "unitarian trick," which is to say he introduces the compact form of a given semisimple Lie algebra and uses integration on the corresponding compact group G. (This trick was already described in §9.3, in the context of proving complete reducibility of representations of a semisimple algebra.)

Indeed, the main reason for including this section (which is, after all, logically unnecessary) is to acquaint the reader with the "classical" treatment of Lie groups via their compact forms. This treatment follows very much the same lines as the representation theory of finite groups. To begin with, we replace the average $(1/|G|)\sum_{g \in G} f(g)$ by the integral $\int_G f(g)\, d\mu$, the volume element $d\mu$ chosen to be translation invariant and such that $\int_G d\mu = 1$. If $\rho\colon G \to \operatorname{Aut}(V)$ is a finite-dimensional representation, with character

$$\chi_V(g) = \operatorname{Trace}(\rho(g)),$$

then $\int_G \rho(g)\, d\mu \in \operatorname{Hom}(V, V)$ is idempotent, and it is the projection onto the invariant subspace V^G. So $\int_G \chi_V(g)\, d\mu = \dim(V^G)$. Applied to $\operatorname{Hom}(V, W)$ as before, since $\chi_{\operatorname{Hom}(V,W)} = \bar{\chi}_V \chi_W$, it follows that

$$\int_G \bar{\chi}_V \chi_W \, d\mu = \dim(\operatorname{Hom}_G(V, W)).$$

So if V and W are irreducible,

$$\int_G \bar{\chi}_V \chi_W \, d\mu = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

Up to now, everything is completely analogous to the case of finite groups, and is proved in exactly the same way. The last general fact, analogous to the basic Proposition 2.30, is harder in the compact case:

Peter–Weyl Theorem. The characters of irreducible representations span a dense subspace of the space of continuous class functions.

It is, moreover, the case that the coordinate functions of the irreducible matrix representations span a dense subspace of all continuous (or L^2) functions on G. For the proof of these statements we refer to [Ad] or [B-tD]. Given the fundamental role that (2.30) played in the analysis of representations of finite groups, it is not surprising that the Peter–Weyl theorem is the cornerstone of most treatments of compact groups, even though it has played no role so far in this book.

We now proceed to indicate how the original proof of the Weyl character formula went in this setting. In this section, G will denote a fixed compact group, whose Lie algebra $\mathfrak{g}$ is a real form of the semisimple complex Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We have seen that

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{l}_\alpha,$$

compatible with the usual decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$ when complexified. The real Cartan algebra $\mathfrak{h}$ acts by rotations on the planes $\mathfrak{l}_\alpha$.

Now let $T = \exp(\mathfrak{h}) \subset G$. As before we have chosen $\mathfrak{h}$ so that it contains the lattice $2\pi i\Gamma$ which is the kernel of the exponential map from $\mathfrak{h}_{\mathbb{C}}$ to the simply-connected form of $\mathfrak{g}_{\mathbb{C}}$, so $T \cong (S^1)^n$ is a compact torus.

In this compact case we can realize the Weyl group on the group level again:

Claim 26.15. $N(T)/T \cong \mathfrak{W}$.

Proof. For each pair of roots α, $-\alpha$, we have a subalgebra $\mathfrak{s}_\alpha \cong \mathfrak{sl}_2\mathbb{C} \subset \mathfrak{g}_{\mathbb{C}}$, with a corresponding $\mathfrak{su}_2 \subset \mathfrak{g}$. Exponentiating gives a subgroup $\mathrm{SU}(2) \subset G$. The element $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acts by Ad, taking H to $-H$, X to Y, and Y to X. It is in $N(T)$, and, with B as in the preceding section, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp\left(\frac{1}{2}\pi i B\right)$. Then $\exp\left(\frac{1}{2}\pi i B\right) \in \mathfrak{g}$ acts by reflection in the hyperplane $\alpha^\perp \subset \mathfrak{h}$. □

Note that $\mathfrak{W}$ acting on $\mathfrak{h}$ takes the lattice $2\pi i\Gamma$ to itself, so $\mathfrak{W}$ acts on $T = \mathfrak{h}/2\pi i\Gamma$ by conjugation.

Theorem 26.16. *Every element of G is conjugate to an element of T. A general element is conjugate to $|\mathfrak{W}|$ such elements of T.*

Sketch of a proof: Note that G acts by left multiplication on the left coset space $X = G/T$. For any $z \in G$, consider the map $f_z: X \to X$ which takes yT to zyT. The claim is that f_z must have a fixed point, i.e., there is a y such that $y^{-1}zy \in T$. Since all f_z are homotopic, and X is compact, the Lefschetz number of f_z is the topological Euler characteristic of X. The first statement follows from the claim that this Euler characteristic is not zero. This is a good exercise for the classical groups; see [Bor2] for a general proof. For another proof see Remark 26.20 below.

For the second assertion, check first that any element that commutes with every element of T is in T. Take an "irrational" element x in T so that its multiples are dense in T. Then for any $y \in G$, $yxy^{-1} \in T \Leftrightarrow yTy^{-1} = T$, and $yxy^{-1} = x \Leftrightarrow y \in T$. This gives precisely $|\mathfrak{W}|$ conjugates of x that are in T.

Corollary 26.17. *The class functions on G are the $\mathfrak{W}$-invariant functions on T.*

Suppose G is a real form of the complex semisimple group $G_{\mathbb{C}}$, i.e., G is a real analytic closed subgroup of $G_{\mathbb{C}}$, and the Lie algebra of $G_{\mathbb{C}}$ is $\mathfrak{g}_{\mathbb{C}}$. The characters on $G_{\mathbb{C}}$ can be written $\sum n_\mu e^{2\pi i\mu}$, the sum over μ in the weight lattice Λ; they are invariant under the Weyl group. From what we have seen, they can be identified with $\mathfrak{W}$-invariant functions on the torus T. Let us work this out for the classical groups:

Case (A_n): $G = \mathrm{SU}(n+1)$. The Lie algebra $\mathfrak{su}_{n+1}$ consists of skew-Hermitian matrices,

$$\mathfrak{h} = \mathfrak{su}_{n+1} \cap \mathfrak{sl}_{n+1}\mathbb{R} = \{\text{imaginary diagonal matrices of trace } 0\},$$

and $T = \{\mathrm{diag}(e^{2\pi i\vartheta_1}, \ldots, e^{2\pi i\vartheta_{n+1}}): \sum \vartheta_j = 0\}$. In this case, the Weyl group $\mathfrak{W}$ is the symmetric group $\mathfrak{S}_{n+1}$, represented by permutation matrices (with one entry ± 1 on each row and column, other entries 0) modulo T. Let z_i: $T \to S^1$ correspond to the ith diagonal entry $e^{2\pi i\vartheta_i}$. So characters on T are symmetric polynomials in $z_1, \ldots, z_{n+1}$ modulo the relation $z_1 \cdot \ldots \cdot z_{n+1} = 1$. Therefore, characters on $\mathrm{SU}(n+1)$ are symmetric polynomials in $z_1, \ldots, z_{n+1}$.

Case (B_n): $G = \mathrm{SO}(2n+1)$. $\mathfrak{h}$ consists of matrices with n 2×2 blocks of the form

$$\begin{pmatrix} \cos(2\pi\vartheta_i) & -\sin(2\pi\vartheta_i) \\ \sin(2\pi\vartheta_i) & \cos(2\pi\vartheta_i) \end{pmatrix}$$

along the diagonal, and one 1 in the lower right corner. Again we see that $T = (S^1)^n$. This time $N(T)$ will have block permutations to interchange the blocks, and also matrices with some blocks $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in the squares along the diagonal, with the other blocks 2×2 identity matrices, with a ± 1 in the corner to make the determinant positive; these take ϑ_i to $-\vartheta_i$ for each i where a block is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This again realizes the Weyl group as a semidirect product of $\mathfrak{S}_n$ and $(\mathbb{Z}/2)^n$. With z_i identified with $e^{2\pi i\vartheta_i}$ again, we see that the characters are the symmetric polynomials in the variables $z_i + z_i^{-1}$, i.e., in $\cos(2\pi\vartheta_1), \ldots, \cos(2\pi\vartheta_n)$.

Case (D_n): $G = \mathrm{SO}(2n)$. $\mathfrak{h}$ is as in the preceding case, but with no lower corner. Since we have no corner to put a -1 in, there can be only an even number of blocks of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, reflecting the fact that $\mathfrak{W}$ is a semidirect product of $(\mathbb{Z}/2)^{n-1}$ and $\mathfrak{S}_n$. This time the invariants are symmetric polynomials in the $z_i + z_i^{-1}$, and one additional $\Pi_i(z_i - z_i^{-1})$.

Case (C_n): $G = \mathrm{Sp}(2n)$. $\mathfrak{h}$ consists of imaginary diagonal matrices, T consists of diagonal matrices with entries $e^{2\pi i\vartheta_i}$. The Weyl group in generated by permutation matrices and diagonal matrices with entries which are 1's and quaternionic j's: $\mathfrak{W}$ is a semidirect product of $(\mathbb{Z}/2)^n$ and $\mathfrak{S}_n$. The invariants are symmetric polynomials in the $z_i + z_i^{-1}$.

The key to Weyl's analysis is to calculate the integral of a class function f on G as a suitable integral over the torus T. For this, consider the map

$$\pi\colon G/T \times T \to G, \qquad \pi(xT, y) = xyx^{-1}.$$

By what we said earlier, π is a generically finite-sheeted covering, with $|\mathfrak{W}|$ sheets. It follows that

$$\int_G f\,d\mu = \frac{1}{|\mathfrak{W}|}\int_{G/T\times T} \pi^*(f)\pi^* d\mu.$$

Now $\pi^*(f)(xT, y) = f(y)$ since f is a class function. To calculate $\pi^* d\mu$, consider the induced map on tangent spaces

$$\pi_* = d\pi\colon \mathfrak{g}/\mathfrak{h} \times \mathfrak{h} \to \mathfrak{g}.$$

At the point $(x_0 T, y_0) \in G/T \times T$,

$$(x_0 e^{tx}T, y_0 e^{ty}) \mapsto x_0 e^{tx} y_0 e^{ty} e^{-tx} x_0^{-1}.$$

We want to calculate

$$\frac{d}{dt}(x_0 e^{tx} y_0 e^{ty} e^{-tx} x_0^{-1})|_{t=0}(x_0 y_0 x_0^{-1})^{-1},$$

which is

$$x_0(xy_0 + y_0 y - y_0 x)x_0^{-1}(x_0 y_0^{-1} x_0^{-1}) = x_0(x + y_0 y y_0^{-1} - y_0 x y_0^{-1})x_0^{-1}.$$

Now $y_0 y y_0^{-1} = y$ since $y_0 \in T$ and $y \in \mathfrak{h}$. To calculate the determinant of π_* we can ignore the volume-preserving transformation $x_0(\ )x_0^{-1}$. If we identify $\mathfrak{g}$ with $\mathfrak{g}/\mathfrak{h} \times \mathfrak{h}$, the matrix becomes

$$\begin{pmatrix} I - \mathrm{Ad}(y_0) & 0 \\ 0 & I \end{pmatrix}.$$

So the determinant of π_* is $\det(I - \mathrm{Ad}(y_0))$. Now $(\mathfrak{g}/\mathfrak{h})_{\mathbb{C}} = \bigoplus \mathfrak{g}_\alpha$, and $\mathrm{Ad}(y_0)$ acts as $e^{2\pi i\alpha(y_0)}$ on $\mathfrak{g}_\alpha$. Hence

$$\det(\pi_*) = \prod_{\alpha\in R}(1 - e^{2\pi i\alpha}), \tag{26.18}$$

as a function on T alone, independent of the factor G/T. This gives *Weyl's integration formula*:

$$\int_G f\,d\mu_G = \frac{1}{|\mathfrak{W}|}\int_T f(y)\prod_{\alpha\in R}(1 - e^{2\pi i\alpha(y)})\,d\mu_T. \tag{26.19}$$

Remark 26.20. The same argument gives another proof of the theorem that G is covered by conjugates of T. This amounts to the assertion that the map

$\pi: G/T \times T \to G$ of compact manifolds is surjective. By what we saw above, for a generic point $y_0 \in T$ there are exactly $|\mathfrak{W}|$ points in $\pi^{-1}(y_0)$, and at each of these the Jacobian determinant is the same (nonzero) number. It follows that the topological degree of the map π is $|\mathfrak{W}|$, so the map must be surjective.

Now $(1 - e^{2\pi i\alpha})(1 - e^{-2\pi i\alpha}) = (e^{\pi i\alpha} - e^{-\pi i\alpha})(\overline{e^{\pi i\alpha} - e^{-\pi i\alpha}})$, so if we set

$$\Delta = \prod_{\alpha \in R^+} (e^{\pi i\alpha} - e^{-\pi i\alpha}),$$

then $\det(\pi_*) = \Delta\bar{\Delta}$. As we saw in Lemma 24.3, $\Delta = A_\rho$, where ρ is half the sum of the positive roots and, for any weight μ,

$$A_\mu = \sum_{W \in \mathfrak{W}} (-1)^W e^{2\pi i W(\mu)}.$$

Now we can complete the second proof of Weyl's character formula: the character of the representation with highest weight λ is $A_{\lambda+\rho}/A_\rho$. Since we saw in §24.1 that $A_{\lambda+\rho}/A_\rho$ has highest weight λ and (see Corollary 24.6) its value at the identity is positive, it suffices to show that the integral of $\int_G \chi\bar{\chi} = 1$, where $\chi = A_{\lambda+\rho}/A_\rho$. By Weyl's integration formula,

$$\begin{aligned}\int_G \chi\bar{\chi} &= \frac{1}{|\mathfrak{W}|}\int_T \chi\bar{\chi}\Delta\bar{\Delta} = \frac{1}{|\mathfrak{W}|}\int_T A_{\lambda+\rho}\overline{A_{\lambda+\rho}} \\ &= \frac{1}{|\mathfrak{W}|}\int_T \sum_{W \in \mathfrak{W}} (-1)^W e^{2\pi i W(\lambda+\rho)} \cdot \sum_{W \in \mathfrak{W}} (-1)^W e^{-2\pi i W(\lambda+\rho)} = 1,\end{aligned}$$

which concludes the proof.

§26.3. Real, Complex, and Quaternionic Representations

The final topic we want to take up is the classification of irreducible complex representations of semisimple Lie groups or algebras into those of real, quaternionic, or complex type. To define our terms, given a real semisimple Lie group G_0 or its Lie algebra $\mathfrak{g}_0$ and a representation of G_0 or $\mathfrak{g}_0$ on a complex vector space V we say that the representation V is *real*, or of *real type*, if it comes from a representation of G_0 or $\mathfrak{g}_0$ on a real vector space V_0 by extension of scalars ($V = V_0 \otimes_{\mathbb{R}} \mathbb{C}$); this is equivalent to saying that it has a conjugate linear endomorphism whose square is the identity. It is *quaternionic* if it comes from a quaternionic representation by restriction of scalars, or equivalently if it has a conjugate linear endomorphism whose square is minus the identity. Finally, we say that the representation is *complex* if it is neither of these. (Compare with Theorem 3.37 for finite groups.)

Having completely classified the irreducible representations of the classical complex Lie algebras, and having described all the real forms of these Lie algebras, we have a clear-cut problem: to detemine the type of the restriction of each representation to each real form. Rather than try to answer this in every case, however, we will instead mention some of the ideas that allow us to answer this question, and then focus on the cases of the split forms (where the answer is easy) and the compact forms (where the answer is more interesting, and where we have more tools to play with). We assume the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$ is simple, so irreducible representations of $\mathfrak{g}_0$ are restrictions of unique irreducible representations of $\mathfrak{g}$ (cf. (26.14)); in particular, we have the classification of irreducible representations by dominant weights.

To begin with, the tensor products of two real, or two quaternionic, or of a pair of complex conjugate representations is always real; and exterior powers of real and quaternionic representations are equally easy to analyze, as for finite groups (see Exercise 3.43). Such tensor and exterior powers may not be irreducible, but the following criterion can often be used to describe an irreducible component of highest weight that occurs inside them:

Exercise 26.21*. Suppose W is a representation of a semisimple group G that is real or quaternionic, and suppose W has a highest weight λ that occurs with multiplicity 1. Show that the irreducible representations Γ_λ with highest weight λ has the same type as W.

We may apply this in particular to the tensor product $\Gamma_\lambda \otimes \Gamma_\mu$ of the irreducible representations of $\mathfrak{g}$ with highest weights λ and μ; since the irreducible representation $\Gamma_{\lambda+\mu}$ with highest weight $\lambda + \mu$ appears once in this tensor product, we deduce

Exercise 26.22*. (i) If Γ_λ and Γ_μ are both real or both quaternionic, then $\Gamma_{\lambda+\mu}$ is real. (ii) If Γ_λ is real and Γ_μ is quaternionic, then $\Gamma_{\lambda+\mu}$ is quaternionic. (iii) If Γ_λ and Γ_μ are complex and conjugate, then $\Gamma_{\lambda+\mu}$ is real.

The last two exercises almost completely answer the question of the representations of the split forms of the classical groups: we have

Proposition 26.23. *Every irreducible representation of the split forms* $\mathfrak{sl}_{n+1}\mathbb{R}$, $\mathfrak{so}_{n+1,n}\mathbb{R}$, $\mathfrak{sp}_{2n}\mathbb{R}$, *and* $\mathfrak{so}_{n,n}\mathbb{R}$ *of the classical Lie algebras is real.*

Proof. In each of these cases, the standard representation V is real, from which it follows that the exterior powers $\bigwedge^k V$ are real, from which it follows that the symmetric powers $\mathrm{Sym}^{a_k}(\bigwedge^k V)$ are real. Now, in the cases of $\mathfrak{sl}_{n+1}\mathbb{R}$ and $\mathfrak{sp}_{2n}\mathbb{R}$, we have seen that the highest weights ω_k of the representations $\bigwedge^k V$ for $k = 1, \ldots, n$ form a set of fundamental weights: that is, every irreducible representation Γ has highest weight $\sum a_k \cdot \omega_k$ for some non-negative integers $a_1, \ldots, a_n$. It follows that Γ appears once in the tensor product

$$\mathrm{Sym}^{a_1} V \otimes \mathrm{Sym}^{a_2}(\textstyle\bigwedge^2 V) \otimes \cdots \otimes \mathrm{Sym}^{a_n}(\textstyle\bigwedge^n V)$$

and so is real. (Alternatively, Weyl's construction produces real representations when applied to real vector spaces.)

The only difference in the orthogonal case is that some of the exterior powers $\wedge^k V$ of the standard representation must be replaced in this description by the spin representation(s). That the spin representations are real follows from the construction in Lecture 20, cf. Exercise 20.23; the result in this case then follows as before. □

The Compact Case

We turn now to the compact forms of the classical Lie algebras. In this case, the theory behaves very much like that of finite groups, discussed in Lecture 5. Specifically, any action of a compact group G_0 on a complex vector space V preserves a nondegenerate Hermitian inner product (obtained, for example, by choosing one arbitrarily and averaging its translates under the action of G_0). It follows that the dual of V is isomorphic to its conjugate, so that V will be either real or quaternionic exactly when it is isomorphic to its dual V^*. (In terms of characters, this says that the character $\mathrm{Char}(V)$ is invariant under the automorphism of $\mathbb{Z}[\Lambda]$ which takes $e(\mu)$ to $e(-\mu)$; for groups, this says the character is real.) More precisely, an irreducible representation of a compact group/Lie algebra will be real (resp. quaternionic) if and only if it has an invariant nondegenerate symmetric (resp. skew-symmetric) bilinear form. In other words, the classification of an irreducible V is determined by whether

$$V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V$$

contains the trivial representation, and, if so, in which factor. So determining which type a representation belongs to is a very special case of the general plethysm problem of decomposing such representations.

With this said, we consider in turn the algebras $\mathfrak{su}_n$, $\mathfrak{u}_n\mathbb{H}$, and $\mathfrak{so}_m\mathbb{R}$.

Let Γ_λ be the irreducible representation of $\mathfrak{sl}_n\mathbb{C}$ with highest weight $\lambda = \sum a_i \cdot \omega_i$, where $\omega_i = L_1 + \cdots + L_i$, $i = 1, \ldots, n-1$ are the fundamental weights of $\mathfrak{sl}_n\mathbb{C}$. The dual of Γ will have highest weight $\sum a_{n-i} \cdot \omega_i$, so that Γ will be real or quaternionic if and only if $a_i = a_{n-i}$ for all i. We now distinguish three cases:

(i) If n is odd, then the sublattice of weights $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i is freely generated by the sums $\omega_i + \omega_{n-i}$ for $i = 1, \ldots, (n-1)/2$. Now, ω_i is the highest weight of the exterior power $\wedge^i V$, so that the irreducible representation with highest weight $\omega_i + \omega_{n-i}$ will appear once in the tensor product

$$\wedge^i V \otimes \wedge^{n-i} V = (\wedge^i V) \otimes (\wedge^i V)^*,$$

which by Exercise 26.21 above is real. It follows that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i, the irreducible representation Γ_λ is real.

(iia) If $n = 2k$ is even, then the sublattice of weights $\lambda = \sum a_i \cdot \omega_i$ with

$a_i = a_{n-i}$ for all i is freely generated by the sums $\omega_i + \omega_{n-i}$ for $i = 1, \ldots, k-1$, together with the weight ω_k. As before, the irreducible representations with highest weight $\omega_i + \omega_{n-i}$ are all real. Moreover, in case n is divisible by 4 the representation $\wedge^k V$ is real as well, since $\wedge^k V$ admits a symmetric bilinear form

$$\wedge^k V \otimes \wedge^k V \to \wedge^{2k} V = \mathbb{C}$$

given by wedge product. It follows then as before that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i, the irreducible representation Γ_λ is real.

(iib) In case n is congruent to 2 mod 4, the analysis is similar to the last case except that wedge product gives a skew-symmetric bilinear pairing on $\wedge^k V$. The representation $\wedge^k V$ is thus quaternionic, and it follows that for any weight $\lambda = \sum a_i \cdot \omega_i$ with $a_i = a_{n-i}$ for all i, the irreducible representation Γ_λ is real if a_k is even, quaternionic if a_k is odd. In sum, then, we have

Proposition 26.24. *For any weight $\lambda = \sum a_i \cdot \omega_i$ of $\mathfrak{su}_n$, the irreducible representation Γ_λ with highest weight λ is: complex if $a_i \neq a_{n-i}$ for any i; real if $a_i = a_{n-i}$ for all i and n is odd, or $n = 4k$, or $n = 4k + 2$ and a_{2k+1} is even; and quaternionic if $a_i = a_{n-i}$ for all i and $n = 4k + 2$ and a_{2k+1} is odd.*

Next, we consider the case of the compact form $\mathfrak{u}_n\mathbb{H}$ of $\mathfrak{sp}_{2n}\mathbb{C}$. To begin with, we note that since the restriction to $\mathfrak{u}_n\mathbb{H}$ of the standard representation of $\mathfrak{sp}_{2n}\mathbb{C}$ on $V \cong \mathbb{C}^{2n}$ is quaternionic, the exterior power $\wedge^k V$ is real for k even and quaternionic for k odd. Since the highest weights ω_k of $\wedge^k V$ for $k = 1, \ldots, n$ form a set of fundamental weights, this completely determines the type of the irreducible representations of $\mathfrak{u}_n\mathbb{H}$: we have

Proposition 26.25. *For any weight $\lambda = \sum a_i \cdot \omega_i$ of $\mathfrak{u}_n\mathbb{H}$, the irreducible representation Γ_λ with highest weight λ is real if a_i is even for all odd i, and quaternionic if a_i is odd for any odd i.*

Next, we consider the odd orthogonal algebras. Part of this is easy: since the restriction to $\mathfrak{so}_{2n+1}\mathbb{R}$ of the standard representation V of $\mathfrak{so}_{2n+1}\mathbb{C}$ is real, so are all its exterior powers; and it follows that any representation of $\mathfrak{so}_{2n+1}\mathbb{R}$ whose highest weight lies in the sublattice of index two generated by the highest weights of these exterior powers is real. It remains, then, to describe the type of the spin representation; the answer, whose verification we leave as Exercise 26.28 below, is that the spin representation Γ_α of $\mathfrak{so}_{2n+1}\mathbb{C}$ (that is, the irreducible representation whose highest weight is one-half the highest weight of $\wedge^n V$) is real when $n \equiv 0$ or 3 mod 4, and quaternionic if $n \equiv 1$ or 2 mod 4. This yields

Proposition 26.26. *Let ω_i be the highest weight of the representation $\wedge^i V$ of $\mathfrak{so}_{2n+1}\mathbb{C}$. For any weight $\lambda = a_1\omega_1 + \cdots + a_{n-1}\omega_{n-1} + a_n\omega_n/2$ of $\mathfrak{so}_{2n+1}\mathbb{R}$, the irreducible representation Γ_λ with highest weight λ is real if a_n is even, or if n is* *congruent to* 0 *or* 3 mod 4; *if* a_n *is odd and* $n \equiv 1$ *or* 2 mod 4, *then* Γ_λ *is quaternionic.*

(Note that, in each of the last two cases, the fact that every representation is either real or quaternionic follows from the observation that the Weyl group action on the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ includes multiplication by -1.)

Finally, we have the even orthogonal Lie algebras. As before, the exterior powers of the standard representation V are all real, but we now have two spin representations to deal with, with highest vectors (in the notation of Lecture 19) $\alpha = (L_1 + \cdots + L_n)/2$ and $\beta = (L_1 + \cdots + L_{n-1} - L_n)/2$. The first question is whether these two are self-conjugate or conjugate to each other. In case n is even, as in the case of the symplectic and odd orthogonal algebras, the Weyl group action on the Cartan subalgebra contains multiplication by -1 (the Weyl group contains the automorphism of $\mathfrak{h}^*$ reversing the sign of any even number of the basis elements L_i), so that Γ_α and Γ_β will be isomorphic to their duals; if n is odd, on the other hand, we see that Γ_α will have $-\beta$ as a weight, so that Γ_α and Γ_β will be complex representations dual to each other. We consider these cases in turn.

(i) Suppose first that n is odd, and say λ is any weight, written as

$$\lambda = a_1\omega_1 + \cdots + a_{n-2}\omega_{n-2} + a_{n-1}\beta + a_n\alpha.$$

If $a_{n-1} \neq a_n$, the representation Γ_λ with highest weight λ will not be isomorphic to its dual, and so will be complex. On the other hand, $\Gamma_{\alpha+\beta}$ appears once in $\Gamma_\alpha \otimes \Gamma_\beta = \mathrm{End}(\Gamma_\alpha)$, and so is real; thus, if $a_{n-1} = a_n$, the representation Γ_λ will be real.

(ii) If, by contrast, n is even then all representations of $\mathfrak{so}_{2n}\mathbb{R}$ will be either real or quaternionic. The half-spin representations Γ_α and Γ_β are real if $n \equiv 0$ (mod 4), quaternionic if $n \equiv 2$ (mod 4), a fact that we leave as Exercise 26.28. It follows that, with λ as above, Γ_λ will be real if either n is divisible by 4, or if $a_{n-1} + a_n$ is even; if $n \equiv 2$ mod 4 and $a_{n-1} + a_n$ is odd, Γ_λ will be quaternionic. In sum, then, we have

Proposition 26.27. *The representation* Γ_λ *of* $\mathfrak{so}_{2n}\mathbb{R}$ *with highest weight* $\lambda = a_1\omega_1 + \cdots + a_{n-2}\omega_{n-2} + a_{n-1}\beta + a_n\alpha$ *will be complex if* n *is odd and* $a_{n-1} \neq a_n$; *it will be quaternionic if* $n \equiv 2$ mod 4 *and* $a_{n-1} + a_n$ *is odd; and it will be real otherwise.*

Exercise 26.28*. Verify the statements made above about the types of the spin representation Γ_α of the orthogonal Lie algebras, i.e., that the spin representation Γ_α of $\mathfrak{so}_{2n+1}\mathbb{R}$ is real when $n \equiv 0$ or 3 (mod 4); and quaternionic if $n \equiv 1$ or 2 (mod 4), and that the half-spin representations of $\mathfrak{so}_{2n}\mathbb{R}$ are real if $n \equiv 0$ (mod 4) and quaternionic if $n \equiv 2$ (mod 4). Show, in fact, that the even Clifford algebras $C_m^{\mathrm{even}} \subset C_m = C(0, m)$ are products of one or two copies of matrix algebras over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, with $\mathbb{R}$ occurring for $m \equiv 0$ or ± 1 mod 8, $\mathbb{C}$ occurring for $m \equiv \pm 2$ mod 8, and $\mathbb{H}$ for $m \equiv \pm 3$ or 4 mod 8.

Exercise 26.29. Show that for a representation V of a compact group G,

$$\int_G \chi_V(g^2) = \begin{cases} 0 & \text{if } V \text{ is complex} \\ 1 & \text{if } V \text{ is real} \\ -1 & \text{if } V \text{ is quaternionic.} \end{cases}$$

Exercise 26.30*. Show that for a representation V of a compact group, the number of irreducible real components it contains, minus the number of quaternionic representations, is the number of times the trivial representation occurs in $\psi^2 V$ in the representation ring, where ψ^2 is the Adams operation (cf. Exercise 23.39).

APPENDICES

These appendices contain proofs of some of the general Lie algebra facts that were postponed during the course, as well as some results from algebra and invariant theory which were used particularly in the "Weyl construction–Schur functor" descriptions of representations.

The first appendix is a fairly serious excursion in polynomial algebra. It proves some basic facts about symmetric functions, especially the Schur polynomials, which occur as characters of representations of GL_n or SL_n, and gives determinantal formulas for them in terms of other basic symmetric polynomials. The last section of Appendix A includes some new identities among symmetric polynomials, which, when the variables are specialized, express characters of representations of Sp_{2n} and SO_m as determinants in the characters of basic representations.

Appendix B gives a short summary of some basic multilinear facts about exterior and symmetric powers. The first two sections can be used as a reference for the conventions and notations we have followed; the third contains a general discussion of constructions such as contractions, many special cases of which were discussed in the main text.

The next three appendices conclude our discussion of the theory of Lie algebras, which began in Lectures 9, 14, and 21. Proofs are given, by standard methods, of the promised general results on semisimplicity, the theorem on conjugacy of Cartan subalgebras, facts about the Weyl group, Ado's theorem that every Lie algebra has a faithful representation, and Levi's theorem that splits the map from a Lie algebra to its semisimple quotient.

The last appendix develops just enough classical invariant theory to find the polynomial invariants for $SL_n\mathbb{C}$, $Sp_{2n}\mathbb{C}$, and $SO_n\mathbb{C}$. This was the key to our proof that Weyl's construction gives the irreducible representations of the symplectic and orthogonal groups.

APPENDIX A

On Symmetric Functions

§A.1. Basic Symmetric Polynomials and Relations among Them

The vector space of homogeneous symmetric polynomials of degree d in k variables $x_1, \ldots, x_k$ has several important bases, usually indexed by the partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0)$ of d into at most k parts, or by Young diagrams with at most k rows (see §4.1). We list four of these bases, which are all valid for polynomials with integer coefficients, or coefficients in any commutative ring.

First we have the monomials in the complete symmetric polynomials:

$$H_\lambda = H_{\lambda_1} \cdot H_{\lambda_2} \cdot \ldots \cdot H_{\lambda_k}, \tag{A.1}$$

where H_j is the jth *complete symmetric polynomial*, i.e., the sum of all distinct monomials of degree j; equivalently,

$$\prod_{i=1}^{k} \frac{1}{1 - x_i t} = \sum_{j=0}^{\infty} H_j t^j.$$

For example, with three variables,

$$H_{(1,1)} = (x_1 + x_2 + x_3)^2,$$
$$H_{(2,0)} = x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3.$$

Next are the *monomial symmetric polynomials*:

$$M_\lambda = \sum X^\alpha, \tag{A.2}$$

the sum over all distinct permutations $\alpha = (\alpha_1, \ldots, \alpha_k)$ of $(\lambda_1, \ldots, \lambda_k)$; here $X^\alpha = x_1^{\alpha_1} \cdot \ldots \cdot x_k^{\alpha_k}$. For example,

$$M_{(1,1)} = x_1x_2 + x_1x_3 + x_2x_3,$$
$$M_{(2,0)} = x_1^2 + x_2^2 + x_3^2.$$

The third are the monomials in the elementary symmetric functions. Unlike the first two, these are parametrized by partitions μ of d in integers no larger than k, i.e., $k \geq \mu_1 \geq \cdots \geq \mu_l \geq 0$. These are exactly the partitions that are conjugate to a partition of d into at most k parts. (The *conjugate* to a partition λ is the partition whose Young diagram is obtained from that of λ by interchanging rows and columns. We denote the conjugate of λ by λ', although the notation $\tilde{\lambda}$ is also common.) For such μ set

$$E_\mu = E_{\mu_1} \cdot E_{\mu_2} \cdot \ldots \cdot E_{\mu_l}, \tag{A.3}$$

where E_j is the jth *elementary symmetric polynomial*, i.e.,

$$E_j = \sum_{i_1 < \cdots < i_j} x_{i_1} \cdot \ldots \cdot x_{i_j}, \qquad \prod_{i=1}^{k} (1 + x_i t) = \sum_{j=0}^{\infty} E_j t^j.$$

For example,

$$E_{(1,1)} = (x_1 + x_2 + x_3)^2,$$
$$E_{(2,0)} = x_1x_2 + x_1x_3 + x_2x_3.$$

The fourth are the *Schur polynomials*, which may be the most important, although they are less often met in modern algebra courses:

$$S_\lambda = \frac{|x_j^{\lambda_i + k - i}|}{|x_j^{k-i}|} = \frac{|x_j^{\lambda_i + k - i}|}{\Delta}, \tag{A.4}$$

where $\Delta = \prod_{i<j}(x_i - x_j)$ is the discriminant, and $|a_{i,j}|$ denotes the determinant of a $k \times k$ matrix. For example,

$$S_{(1,1)} = x_1x_2 + x_1x_3 + x_2x_3,$$
$$S_{(2,0)} = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3.$$

The first task of this appendix is to describe some relations among these symmetric polynomials. For example, one sees quickly that

$$S_{(1,1)} = E_{(2,0)} = H_1^2 - H_2,$$
$$S_{(2,0)} = H_{(2,0)} = E_1^2 - E_2,$$
$$S_{(1,0)} \cdot S_{(1,0)} = S_{(1,1)} + S_{(2,0)}.$$

These are special cases of three important formulas involving Schur polynomials, which we state next. The first two are known as *determinantal* *formulas*. The first is also known as the *Jacobi–Trudy identity*. From geometry, the first two are sometimes called *Giambelli's formulas*, and the third is *Pieri's formula*. The proofs will be given in the next section.

$$S_\lambda = |H_{\lambda_i+j-i}| = \begin{vmatrix} H_{\lambda_1} & H_{\lambda_1+1} \cdots H_{\lambda_1+k-1} \\ H_{\lambda_2-1} & H_{\lambda_2} \cdots & \\ \vdots & & \\ H_{\lambda_k-k+1} \cdots & & H_{\lambda_k} \end{vmatrix}. \tag{A.5}$$

Note that if $\lambda_{p+1} = \cdots = \lambda_k = 0$, the determinant on the right is the same as the determinant of the upper left $p \times p$ corner. The second is

$$S_\lambda = |E_{\mu_i+j-i}| = \begin{vmatrix} E_{\mu_1} & E_{\mu_1+1} \cdots E_{\mu_1+l-1} \\ E_{\mu_2-1} & E_{\mu_2} \cdots & \\ \vdots & & \\ E_{\mu_l-l+1} \cdots & & E_{\mu_l} \end{vmatrix}, \tag{A.6}$$

where $\mu = (\mu_1, \dots, \mu_l)$ is the conjugate partition to λ.

The third "Pieri" formula tells how to multiply a Schur polynomial S_λ by a basic Schur polynomial $S_{(m)} = H_m$[1]:

$$S_\lambda S_{(m)} = \sum S_\nu, \tag{A.7}$$

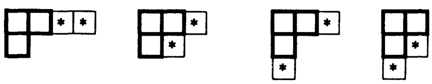
the sum over all ν whose Young diagram can be obtained from that of λ by adding a total of m boxes to the rows, but with no two boxes in the same column, i.e., those $\nu = (\nu_1, \dots, \nu_k)$ with

$$\nu_1 \geq \lambda_1 \geq \nu_2 \geq \lambda_2 \geq \cdots \geq \nu_k \geq \lambda_k \geq 0,$$

and $\sum \nu_j = \sum \lambda_j + m = d + m$. For example, the identity

$$S_{(2,1)} \cdot S_{(2)} = S_{(4,1)} + S_{(3,2)} + S_{(3,1,1)} + S_{(2,2,1)}$$

can be seen from the pictures



One can use the Pieri and determinantal formulas to multiply any two Schur polynomials, but there is a more direct formula, which generalizes Pieri's formula. This *Littlewood–Richardson rule* gives a combinatorial formula for the coefficients $N_{\lambda\mu\nu}$ in the expansion of a product as a linear combination of Schur polynomials:

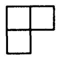
[1] When k is fixed, we often omit zeros at the end of partitions, so (m) denotes the partition $(m, 0, \dots, 0)$.

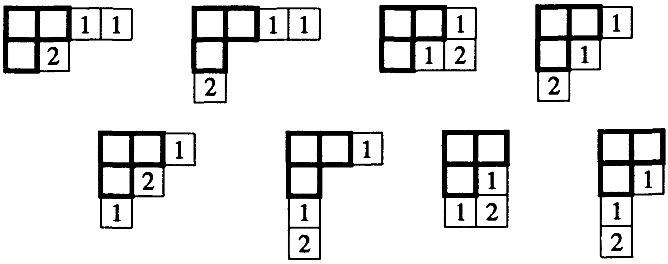
$$S_\lambda \cdot S_\mu = \sum N_{\lambda\mu\nu} S_\nu. \tag{A.8}$$

Here λ is a partition of d, μ a partition of m, and the sum is over all partitions ν of $d + m$ (each with at most k parts). The Littlewood–Richardson rule says that *$N_{\lambda\mu\nu}$ is the number of ways the Young diagram for λ can be expanded to the Young diagram for ν by a strict μ-expansion.* If $\mu = (\mu_1, \ldots, \mu_k)$, a *$\mu$-expansion* of a Young diagram is obtained by first adding μ_1 boxes, according to the above description in Pieri's formula, and putting the integer 1 in each of these μ_1 boxes; then adding similarly μ_2 boxes with a 2, continuing until finally μ_k boxes are added with the integer k. The expansion is called *strict* if, when the integers in the boxes are listed from right to left, starting with the top row and working down, and one looks at the first t entries in this list (for any t between 1 and $\mu_1 + \cdots + \mu_k$), each integer p between 1 and $k - 1$ occurs at least as many times as the next integer $p + 1$.

For example, the equation

$$S_{(2,1)} \cdot S_{(2,1)} = S_{(4,2)} + S_{(4,1,1)} + S_{(3,3)} + 2S_{(3,2,1)} + S_{(3,1,1,1)} + S_{(2,2,2)} + S_{(2,2,1,1)}$$

can be seen by listing the strict (2, 1)-expansions of the Young diagram :



A proof of the Littlewood–Richardson rule can be found in [Mac, §I.9]; for the other results of this appendix we can get by without using it.

Formula (A.7), applied inductively, yields

$$H_\lambda = S_{(\lambda_1)} \cdot S_{(\lambda_2)} \cdot \ldots \cdot S_{(\lambda_k)} = \sum K_{\mu\lambda} S_\mu, \tag{A.9}$$

where *$K_{\mu\lambda}$ is the number of ways one can fill the boxes of the Young diagram of μ with λ_1* 1's, *λ_2* 2's, *up to λ_k* k's, *in such a way that the entries in each row are nondecreasing, and those in each column are strictly increasing.* Such a tableau is called a *semistandard tableau on μ of type λ*. These integers $K_{\mu\lambda}$ are all non-negative, with

$$K_{\lambda\lambda} = 1 \quad \text{and} \quad K_{\mu\lambda} = 0 \quad \text{if } \lambda > \mu, \tag{A.10}$$

i.e., if the first nonvanishing $\lambda_i - \mu_i$ is positive; in addition, $K_{\mu\lambda} = 0$ if λ has more nonzero terms than μ. For example, if $k = 3$, $(K_{\mu\lambda})$ is given by the matrix

	1	1	1
	0	1	2
	0	0	1

The integers $K_{\mu\lambda}$ are called *Kostka numbers.*

Exercise A.11. Show that $K_{\mu\lambda}$ is nonzero if and only if

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i \leq \mu_1 + \mu_2 + \cdots + \mu_i$$

for all $i \geq 1$.

When $\lambda = (1, 1, \ldots, 1)$, $K_{\mu(1,\ldots,1)}$ *is the number of standard tableaux on the diagram of* μ, where a *standard tableau* is a numbering of the d boxes of a Young diagram by the integers 1 through d, increasing in both rows and columns.

We need one more formula involving Schur polynomials, which comes from an identity of Cauchy. Let $y_1, \ldots, y_k$ be another set of indeterminates, and write $P(x)$ and $P(y)$ for the same polynomial P expressed in terms of variables $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$, respectively. The formula we need is

$$\det\left|\frac{1}{1 - x_i y_j}\right| = \frac{\Delta(x)\Delta(y)}{\prod_{i,j}(1 - x_i y_j)}. \tag{A.12}$$

The proof is by induction on k. To compute the determinant, first subtract the first row from each of the other rows, noting that

$$\frac{1}{1 - x_i y_j} - \frac{1}{1 - x_1 y_j} = \frac{x_i - x_1}{1 - x_1 y_j} \cdot \frac{y_j}{1 - x_i y_j}$$

and factor out common factors. Then subtract the first column from each of the other columns, this time using the equation

$$\frac{y_j}{1 - x_i y_j} - \frac{y_1}{1 - x_i y_1} = \frac{y_j - y_1}{1 - x_i y_1} \cdot \frac{1}{1 - x_i y_j}$$

to factor out common factors. One is left with a matrix whose first row is $(1\ 0 \ldots 0)$, and whose lower right square has the original entries. The formula follows by induction (cf. [Wel, p. 202]). □

Another form of Cauchy's identity is

$$\frac{1}{\prod_{i,j}(1 - x_i y_j)} = \sum_\lambda S_\lambda(x) S_\lambda(y), \tag{A.13}$$

the sum over all partitions λ with at most k terms. To prove this, expand the determinant whose i, j entry is $(1 - x_i y_j)^{-1} = 1 + x_i y_j + x_i^2 y_j^2 + \cdots$. One sees that for any $l_1 > \cdots > l_k$ the coefficient of $y_1^{l_1} y_2^{l_2} \cdot \ldots \cdot y_k^{l_k}$ is the determinant $|x_j^{l_i}|$. By symmetry of the x and y variables we have

$$\det\left|\frac{1}{1 - x_i y_j}\right| = \sum_l |x_j^{l_i}| \cdot |y_j^{l_i}|. \tag{A.14}$$

Combining (A.12) with (A.4) gives (A.13). □

Expansion of the left-hand side of (A.13) gives

$$\frac{1}{\prod_{i,j}(1 - x_i y_j)} = \prod_j \left(\sum_{m=0}^{\infty} H_m(x) y_j^m\right) = \sum_\lambda H_\lambda(x) M_\lambda(y). \tag{A.15}$$

Since the polynomials H_λ as well as the M_μ form a basis for the symmetric polynomials, one can define a bilinear form $\langle\ ,\ \rangle$ on the space of homogeneous symmetric polynomials of degree d in k variables, by requiring that

$$\langle H_\lambda, M_\mu \rangle = \delta_{\lambda,\mu}, \tag{A.16}$$

where $\delta_{\lambda,\mu}$ is 1 if $\lambda = \mu$ and 0 otherwise. The basic fact here is that *the Schur polynomials form an orthonormal basis for this pairing*:

$$\langle S_\lambda, S_\mu \rangle = \delta_{\lambda,\mu}. \tag{A.17}$$

In particular, this implies that the pairing $\langle\ ,\ \rangle$ is *symmetric*. Equation (A.17) is easily deduced from the preceding equations, as follows. Write $S_\lambda = \sum a_{\lambda\gamma} H_\gamma = \sum b_{\gamma\lambda} M_\gamma$, for some integer matrices $a_{\lambda\gamma}$ and $b_{\gamma\lambda}$. Then

$$\langle S_\lambda, S_\mu \rangle = \sum_\gamma a_{\lambda\gamma} b_{\gamma\mu}. \tag{A.18}$$

In order that

$$\sum_\lambda S_\lambda(x) S_\lambda(y) = \sum_{\lambda,\gamma,\rho} a_{\lambda\gamma} H_\gamma(x) b_{\rho\lambda} M_\rho(y)$$

be equal to $\sum_\gamma H_\gamma(x) M_\gamma(y)$, which it must by (A.13) and (A.15), we must have

$$\sum_\lambda b_{\rho\lambda} a_{\lambda\gamma} = \delta_{\rho,\gamma}.$$

This is equivalent to the equation $\sum_\gamma a_{\lambda\gamma} b_{\gamma\mu} = \delta_{\lambda,\mu}$, which by (A.18) implies (A.17).

Because of this duality, formula (A.9) is equivalent to the equation

$$S_\mu = \sum_\lambda K_{\mu\lambda} M_\lambda. \tag{A.19}$$

This gives another formula for these Kostka numbers: $K_{\mu\lambda}$ *is the coefficient of* X^λ *in* S_μ, where $X^\lambda = x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$.

The identities (A.9) and (A.19) for the basic symmetric polynomials allow us to relate the coefficients of X^λ in any symmetric polynomial P with the coefficients expanding P as a linear combination of the Schur polynomials. If P is any homogeneous symmetric polynomial of degree d in k variables, and λ is any partition of d into at most k parts, define numbers $\psi_\lambda(P)$ and $\omega_\lambda(P)$ by

$$\psi_\lambda(P) = [P]_\lambda, \tag{A.20}$$

where $[P]_\lambda$ denotes the coefficient of $X^\lambda = x_1^{\lambda_1} \cdot \ldots \cdot x_k^{\lambda_k}$ in P, and

$$\omega_\lambda(P) = [\Delta \cdot P]_l, \quad l = (\lambda_1 + k - 1, \lambda_2 + k - 2, \ldots, \lambda_k); \tag{A.21}$$

here $\Delta = \prod_{i<j}(x_i - x_j)$. We want to compare these two collections of numbers, as λ varies over the partitions.

The first numbers $\psi_\lambda(P)$ are the coefficients in the expression

$$P = \sum \psi_\lambda(P) M_\lambda \tag{A.22}$$

for P as a linear combination of the monomial symmetric polynomials M_λ. The integers $\omega_\lambda(P)$ have a similar interpretation in terms of Schur polynomials:

$$P = \sum \omega_\lambda(P) S_\lambda. \tag{A.23}$$

Note from the definition that the coefficient of X^l in $\Delta \cdot S_\lambda$ is 1, and that no other monomial with strictly decreasing exponents appears in $\Delta \cdot S_\lambda$; from this, formula (A.23) is evident. In this terminology we may rewrite (A.19) and (A.9) as

$$K_{\mu\lambda} = \psi_\lambda(S_\mu) = [S_\mu]_\lambda = \text{coefficient of } X^\lambda \text{ in } S_\mu \tag{A.24}$$

and

$$K_{\mu\lambda} = \omega_\mu(H_\lambda) = [\Delta \cdot H_\lambda]_{(\lambda_1 + k - 1, \ldots, \lambda_k)}. \tag{A.25}$$

Lemma A.26. *For any symmetric polynomial P of degree d in k variables,*

$$\psi_\lambda(P) = \sum_\mu K_{\mu\lambda} \cdot \omega_\mu(P).$$

PROOF. We have

$$\begin{aligned}\sum_\lambda \psi_\lambda(P) M_\lambda = P = \sum_\mu \omega_\mu(P) S_\mu &= \sum_{\lambda,\mu} \omega_\mu(P) K_{\mu\lambda} M_\lambda \\ &= \sum_\lambda \left(\sum_\mu K_{\mu\lambda} \omega_\mu(P) \right) M_\lambda,\end{aligned}$$

and the result follows, since the M_λ are independent. □

We want to apply the preceding discussion when the polynomial P is a product of sums of powers of the variables. Let $P_j = x_1^j + \cdots + x_k^j$, and for

$\mathbf{i} = (i_1, \ldots, i_d)$, a d-tuple of non-negative integers with $\sum \alpha i_\alpha = d$, set

$$P^{(\mathbf{i})} = P_1^{i_1} \cdot P_2^{i_2} \cdot \ldots \cdot P_d^{i_d}.$$

These *Newton* or *power sum* polynomials form a basis for the symmetric functions with rational coefficients, but not with integer coefficients. Let

$$\omega_\lambda(\mathbf{i}) = \omega_\lambda(P^{(\mathbf{i})}).$$

Equivalently,

$$P^{(\mathbf{i})} = \sum \omega_\lambda(\mathbf{i}) S_\lambda. \tag{A.27}$$

For the proof of Frobenius's formula in Lecture 4 we need a formal lemma about these coefficients $\omega_\lambda(\mathbf{i})$:

Lemma A.28. *For partitions λ and μ of d,*

$$\sum_{\mathbf{i}} \frac{1}{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!} \omega_\lambda(\mathbf{i}) \omega_\mu(\mathbf{i}) = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We will use Cauchy's formula (A.13). Note that

$$\log\left(\prod_{i,j} (1 - x_i y_j)^{-1}\right) = \sum_{j=1}^{\infty} \frac{1}{j} P_j(x) P_j(y),$$

so

$$\frac{1}{\prod (1 - x_i y_j)} = \prod_j \exp\left(\frac{1}{j} P_j(x) P_j(y)\right)$$

$$= \sum_{\mathbf{i}} \frac{1}{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!} P^{(\mathbf{i})}(x) P^{(\mathbf{i})}(y)$$

$$= \sum_{\mathbf{i}} \frac{1}{1^{i_1} i_1! \cdot \ldots \cdot d^{i_d} i_d!} \sum_\lambda \omega_\lambda(\mathbf{i}) S_\lambda(x) \sum_\mu \omega_\mu(\mathbf{i}) S_\mu(y).$$

Comparing with (A.13), the conclusion follows. □

Exercise A.29*. Using the pairing $\langle\ ,\ \rangle$ of (A.16), the coefficients $\omega_\lambda(\mathbf{i}) = \omega_\lambda(P^{(\mathbf{i})})$ can be written $\omega_\lambda(\mathbf{i}) = \langle S_\lambda, P^{(\mathbf{i})} \rangle$.

(a) Show that the Newton polynomials are orthogonal for this pairing, and

$$\langle P^{(\mathbf{i})}, P^{(\mathbf{i})} \rangle = 1^{i_1} i_1! 2^{i_2} i_2! \cdot \ldots \cdot d^{i_d} i_d!.$$

Equivalently,

$$S_\lambda = \sum \frac{1}{z(\mathbf{i})} \omega_\lambda(\mathbf{i}) P^{(\mathbf{i})},$$

where the sum is over all partitions $\mathbf{i} = (i_1, \ldots, i_d)$ with $\sum \alpha i_\alpha = d$, and $z(\mathbf{i}) = i_1! 1^{i_1} \cdot i_2! 2^{i_2} \cdot \ldots \cdot i_d! d^{i_d}$.

(b) Show that $\omega_\lambda(\mathbf{i}) = \sum_\nu \langle S_\lambda, M_\nu \rangle \cdot \langle H_\nu, P^{(\mathbf{i})} \rangle$.

We should remark that we have chosen to write our formulas for a fixed number k of variables, since that often simplifies computations when k is small. It is more usual to require the number of variables to be large, at least as large as the numbers being partitioned—or in the limiting ring with an infinite number of variables, cf. Exercise A.32; the formulas for smaller k are then recovered by setting the variables $x_i = 0$ for $i > k$. For example, if $k \geq 2$ we have $S_{(1)}^2 = S_{(2)} + S_{(1,1)}$, which reduces to $S_{(1)}^2 = S_{(2)}$ when $k = 1$.

The next two exercises give formulas for the value of the Schur polynomials when the variables x_i are all set equal to 1; these numbers are the dimensions of the corresponding representations. For a formula for $S_\lambda(1, \ldots, 1)$ involving hook lengths of the Young diagram of λ, see Exercise 6.4.

Exercise A.30*. When $x_i = x^{i-1}$, the numerators in (A.4) are van der Monde determinants, leading to

(i) $$S_\lambda(1, x, x^2, \ldots, x^{k-1}) = x^k \prod_{i<j} \frac{x^{\lambda_i - \lambda_j + j - i} - 1}{x^{j-i} - 1}.$$

Taking the limit as $x \to 1$, one finds

(ii) $$S_\lambda(1, \ldots, 1) = \prod_{i<j} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

By (A.5) and (A.6) we have also the following two formulas:

(iii) $$S_\lambda(1, \ldots, 1) = |h_{\lambda_i + j - i}|, \quad \text{where } \sum h_j t^j = \frac{1}{(1-t)^k}.$$

(iv) $$S_\lambda(1, \ldots, 1) = \left| \binom{k}{\mu_i + j - i} \right|, \quad \text{where } (\mu_1, \ldots, \mu_r) = \lambda'.$$

Exercise A.31*. (a) Show that

$$S_\mu = \sum K_{\mu a} X^a,$$

the sum over all monomials $X^a = x_1^{a_1} \cdot \ldots \cdot x_k^{a_k}$, where, for *any* k-tuple a of non-negative integers, $K_{\mu a}$ is the number of ways to number the boxes of the Young diagram of μ with a_1 1's, a_2 2's, ..., a_k k's, with nondecreasing rows and strictly increasing columns. In particular, the right-hand side is a symmetric polynomial, a fact which is not obvious from the definition.

(b) Deduce that $S_\mu(1, \ldots, 1)$ is the number of ways to number the boxes of the Young diagram of μ with integers from 1 to k, with nondecreasing rows and strictly increasing columns (i.e., the number of semistandard tableaux).

Exercise A.32*. The idea of considering symmetric polynomials in an arbitrarily large number of variables can be formalized by working in the ring $\Lambda = \varprojlim \Lambda(k)$, where $\Lambda(k)$ denotes the ring of symmetric polynomials in k variables. Then

$$\Lambda = \mathbb{Z}[H_1, \ldots H_k, \ldots] = \mathbb{Z}[E_1, \ldots E_k, \ldots]$$

is a graded polynomial ring, with H_i and E_i of degree i. A ring homomorphism $\vartheta\colon \Lambda \to \Lambda$ can be defined by requiring

$$\vartheta(E_i) = H_i \qquad \text{for all } i.$$

(i) Show that ϑ is an involution: $\vartheta^2 = \vartheta$. Equivalently,

$$\vartheta(H_i) = E_i.$$

(ii) If λ' is the conjugate partition to λ, show that

$$\vartheta(S_\lambda) = S_{\lambda'}.$$

(iii) If $P_j = x_1^j + \cdots + x_k^j$ is the jth power sum, show that

$$\vartheta(P_j) = (-1)^{j-1} P_j.$$

(iv) Deduce the formula

$$E_\lambda = \sum_\mu K_{\mu\lambda} \Delta_{\mu'}.$$

(v) Deduce a dual form of (A.7):

$$S_\lambda \cdot S_{(1,\ldots,1)} = S_\lambda \cdot E_m = \sum S_\pi,$$

the sum over all partitions π whose Young diagram can be obtained from that of λ by adding m boxes, with no two in any row.

(vi) Show that

$$H_m = \sum \frac{1}{z(\mathbf{i})} P^{(\mathbf{i})}, \qquad E_m = \sum \frac{(-1)^{\sum(i_j - 1)}}{z(\mathbf{i})} P^{(\mathbf{i})},$$

where the sums are over all $\mathbf{i} = (i_1, \ldots, i_d)$ with $\sum \alpha i_\alpha = d$, and $z(\mathbf{i}) = i_1!1^{i_1} \cdot i_2!2^{i_2} \cdot \ldots \cdot i_d!d^{i_d}$. Note that

$$\sum (-1)^i H_i t^i = (\sum E_i t^i)^{-1}.$$

§A.2. Proofs of the Determinantal Identities

To prove the Jacobi–Trudi identity (A.5), note the identities

$$x_j^p - E_1 x_j^{p-1} + E_2 x_j^{p-1} - \cdots + (-1)^k E_k x_j^{p-k} = 0, \tag{A.33}$$

for any $1 \le j \le k$, $p \ge k$. And for any $0 \le m < k$ and $p \ge k$,

$$H_{p-m} - E_1 H_{p-m-1} + E_2 H_{p-m-2} + \cdots + (-1)^k E_k H_{p-m-k} = 0. \tag{A.34}$$

Both of these follow immediately from the defining power series for the E_j and H_j. Since these two recursion relations are the same, there are universal polynomials $A(p, q)$ in the variables $E_1, \ldots, E_k$ such that

$$\begin{aligned} x_j^p &= A(p, 1)x_j^{k-1} + A(p, 2)x_j^{k-2} + \cdots + A(p, k), \\ H_{p-m} &= A(p, 1)H_{k-m-1} + A(p, 2)H_{k-m-2} + \cdots + A(p, k)H_{-m}. \end{aligned} \tag{A.35}$$

For any integers $\lambda_1, \ldots, \lambda_k$ this leads to matrix identities

$$\begin{aligned} (x_j^{\lambda_i+k-i})_{ij} &= (A(\lambda_i + \mathrm{k} - \mathrm{i}, r))_{ir} \cdot (x_j^{k-r})_{rj}, \\ (H_{\lambda_i+j-i})_{ij} &= (A(\lambda_i + \mathrm{k} - \mathrm{i}, r))_{ir} \cdot (H_{j-r})_{rj}, \end{aligned} \tag{A.36}$$

where $(\ )_{pq}$ denotes the $k \times k$ matrix whose p, q entry is specified between the parentheses. The relations (A.34) also imply:

Lemma (A.37). *The matrices* (H_{q-p}) *and* $((-1)^{q-p}E_{q-p})$ *are lower-triangular matrices with* 1's *along the diagonal, and are inverses of each other.*

The identities (A.36) therefore combine to give

$$(x_j^{\lambda_i+k-i})_{ij} = (H_{\lambda_i+p-i})_{ip} \cdot ((-1)^{q-p}E_{q-p})_{pq} \cdot (x_j^{k-q})_{qj} \tag{A.38}$$

Taking determinants gives (A.5), since the determinant of the matrix in the middle is 1.

Exercise A.39*. Prove the identity

$$|x_j^{l_i}| \cdot \prod_{j=1}^{k} (1 - x_j)^{-1} = \sum |x_j^{m_i}|,$$

the sum over all k-tuples $(m_1, \ldots, m_k)$ of non-negative integers with $m_1 \geq l_1 > m_2 \geq \cdots > m_k \geq l_k$, and deduce Pieri's formula (A.7).

To complete the proofs of the assertions in §A.1, we show that the two determinants appearing in the Giambelli formulas (A.5) and (A.6) are equal, i.e., if $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ are conjugate partitions, then

$$|H_{\lambda_i+j-i}| = |E_{\mu_i+j-i}|. \tag{A.40}$$

Here the H_i and E_i can be any elements (in a commutative ring) satisfying the identity $(\sum H_i t^i) \cdot (\sum (-1)^i E_i t^i) = 1$, with $H_0 = E_0 = 1$ and $H_i = E_i = 0$ for $i < 0$. To prove it, we need a combinatorial characterization of the conjugacy of partitions:

Exercise A.41*. For $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ conjugate partitions, show that the sets

$$\{\lambda_i + n + 1 - i : 1 \leq i \leq k\} \quad \text{and} \quad \{n + j - \mu_j : 1 \leq j \leq l\}$$

form a disjoint union of the set $\{1, \ldots, k + l\}$.

We also need a basic matrix identity which relates minors of a matrix to minors of its inverse (or matrix of cofactors). If $A = (a_{ij})$ is an $r \times r$ matrix, and $S = (s_1, \ldots, s_k)$ and $T = (t_1, \ldots, t_k)$ are two sequences of k distinct integers from $\{1, \ldots, r\}$, let $A_{S,T}$ denote the corresponding minor: $A_{S,T}$ is the determinant of the $k \times k$ matrix whose i, j entry is a_{s_i, t_j}.

Lemma A.42. *Let A and B be $r \times r$ matrices whose product is a scalar matrix $c \cdot I_r$. Let (S, S') and (T, T') be permutations of the sequence $(1, \ldots, r)$, where S and T consists of k integers, S' and T' of $r - k$. Then*

$$c^{r-k} \cdot A_{S,T} = \varepsilon \cdot \det(A) \cdot B_{T',S'},$$

where ε is the product of the signs of the two permutations.

PROOF. By permuting the rows and columns of A, multiplying on the left and right by permutation matrices P and Q corresponding to the two permutations of $(1, \ldots, r)$, we may take the (S, T) minor to the upper left corner:

$$PAQ = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad A_{S,T} = \det A_1.$$

Then

$$Q^{-1}BP^{-1} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \qquad B_{T',S'} = \det B_4.$$

Now taking determinants in the identity

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \cdot \begin{pmatrix} I_k & B_2 \\ 0 & B_4 \end{pmatrix} = \begin{pmatrix} A_1 & 0 \\ A_3 & cI_{r-k} \end{pmatrix}$$

gives the equation $\det(PAQ) \cdot \det(B_4) = \det(A_1) \cdot c^{r-k}$. Since ε is the product of the determinants of P and Q, the lemma follows. □

PROOF OF (A.40). Apply the lemma to $A = (H_{q-p})$ and $B = ((-1)^{q-p} E_{q-p})$, with $r = k + l$, and

$$\begin{aligned} S &= (\lambda_1 + k, \lambda_2 + k - 1, \ldots, \lambda_k + 1), \\ S' &= (k + 1 - \mu_1, k + 2 - \mu_2, \ldots, k + l - \mu_l), \\ T &= (k, k - 1, \ldots, 1), \\ T' &= (k + 1, k + 2, \ldots, k + l). \end{aligned}$$

Then

$$A_{S,T} = \det(H_{(\lambda_i + k + 1 - i) - (k + 1 - j)}) = |H_{\lambda_i + j - i}|.$$

Similarly,

$$\begin{aligned} B_{T',S'} &= |(-1)^{\mu_j + i - j} E_{\mu_j + i - j}| = (-1)^{\sum (\mu_j - j)} (-1)^{\sum i} |E_{\mu_j + i - j}| \\ &= (-1)^d |E_{\mu_j + i - j}|, \end{aligned}$$

with $d = \sum \mu_j = \sum \lambda_i$. Since $\varepsilon = (-1)^d$, (A.40) follows. □

§A.3. Other Determinantal Identities

In this final section we prove some variations of these formulas which are useful for calculating characters of symplectic and orthogonal groups. We want to compare minors, not of $H = (H_{i-j})$ and $E = ((-1)^{i-j}E_{i-j})$, but of matrices H^+ and E^- constructed from them by the following procedures:

For an $r \times r$ matrix $H = (H_{i,j})$, and a fixed integer k between 1 and r, H^+ denotes the $r \times r$ matrix obtained from H by folding H along the kth column, and adding each column to the right of the kth column to the column the same distance to the left. That is,

$$H^+_{i,j} = \begin{cases} H_{i,j} + H_{i,2k-j} & \text{if } j < k \\ H_{i,j} & \text{if } j \geq k \end{cases}$$

(with the convention that $H_{p,q} = 0$ if p or q is not between 1 and r). The matrix E^- is obtained by folding E along its kth row, and subtracting rows above this row from those below:

$$E^-_{i,j} = \begin{cases} E_{i,j} - E_{2k-i,j} & \text{if } i > k \\ E_{i,j} & \text{if } i \leq k. \end{cases}$$

Lemma A.43. *If H and E are lower-triangular matrices with 1's along the diagonal, that are inverse to each other, then the same is true for H^+ and E^-.*

Proof. This is a straightforward calculation: the i, j entry of the matrix $H^+ \cdot E^-$ is

$$\sum_{p=1}^{k-1} (H_{i,p} + H_{i,2k-p})E_{p,j} + H_{i,k}E_{k,j} + \sum_{p=k+1}^{r} H_{i,p}(E_{p,j} - E_{2k-p,j})$$
$$= \sum_{p=1}^{r} H_{i,p}E_{p,j} + \sum_{p=1}^{k-1} H_{i,2k-p}E_{p,j} - \sum_{q=k+1}^{r} H_{i,q}E_{2k-q,j}.$$

The first sum is $\delta_{i,j}$, and the others cancel term by term. □

Proposition A.44. *Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ and $\mu = (\mu_1, \ldots, \mu_l)$ be conjugate partitions. Set*

$$E'_i = E_i \quad \textit{for } i \leq 1, \qquad \textit{and} \qquad E'_i = E_i - E_{i-2} \quad \textit{for } i \geq 2.$$

Then the determinant of the $k \times k$ matrix whose ith *row is*

$$(H_{\lambda_i-i+1} \quad H_{\lambda_i-i+2} + H_{\lambda_i-i} \quad H_{\lambda_i-i+3} + H_{\lambda_i-i-1} \quad \cdots \quad H_{\lambda_i-i+k} + H_{\lambda_i-i-k+2})$$

is equal to the determinant of the $l \times l$ matrix whose ith *row is*

$$(E'_{\mu_i-i+1} \quad E'_{\mu_i-i+2} + E'_{\mu_i-i} \quad E'_{\mu_i-i+3} + E'_{\mu_i-i-1} \quad \cdots \quad E'_{\mu_i-i+l} + E'_{\mu_i-i-l+2}).$$

Each of these determinants is equal to the determinant

$$|E_{\mu_i-i+j} - E_{\mu_i-i-j}|$$

and to the determinant

$$|H''_{\lambda_i-i+j} - H''_{\lambda_i-i-j}|,$$

where $H''_i = H_i$ *for* $i \leq 1$, *and for* $i \geq 2$

$$H''_i = H_i + H_{i-2} + H_{i-4} + \cdots + \begin{cases} H_1 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even.} \end{cases}$$

Proof. With $H = (H_{i-j})$ and $E = ((-1)^{q-p}E_{q-p})$ we can apply the basic lemma (A.42) to the new matrices $A = H^+$ and $B = E^-$, and the same permutations (S, S') and (T, T') used in the proof of (A.40). This time

$$A_{S,T} = \det(H^+_{\lambda_i+k+1-i,k-j+1}),$$

and

$$H^+_{\lambda_i+k+1-i,k-j+1} = \begin{cases} H_{\lambda_i-i+j} + H_{\lambda_i-i-j+2} & \text{if } j = 2, \ldots, k \\ H_{\lambda_i-i+1} & \text{if } j = 1. \end{cases}$$

Similarly,

$$B_{T',S'} = \det(E^-_{k+i,k+j-\mu_j}),$$

with

$$E^-_{k+i,k+j-\mu_j} = (-1)^{\mu_j+i-j}(E_{\mu_j+i+j} - E_{\mu_j+i-j}).$$

As before, Lemma A.42 implies that the determinant of the first displayed matrix of the proposition is equal to that of the third. Noting that

$$E_{\mu_j+i+j} - E_{\mu_j+i-j} = E'_{\mu_j+i+j} + E'_{\mu_j+i+j-2} + \cdots + E'_{\mu_j+i-j+2},$$

one can do elementary column operations on the third matrix, subtracting the first column from the third, then the second by the fourth, etc., to see that the second and third determinants are equal. Since $H_i = H''_i - H''_{i-2}$, the same argument shows the equality of the first and fourth determinants. □

Note that in these four formulas, as in the determinantal formulas for Schur polynomials, if a partition has p nonzero terms, only the upper left $p \times p$ subdeterminant needs to be calculated. We denote by $S_{\langle\lambda\rangle}$ the determinant of the proposition:

$$S_{\langle\lambda\rangle} = |H_{\lambda_i-i+1} \quad H_{\lambda_i-i+2} + H_{\lambda_i-i} \quad \ldots \quad H_{\lambda_i-i+k} + H_{\lambda_i-i-k+2}|. \qquad \text{(A.45)}$$

Dually, set $H'_i = H_i - H_{i-2}$ and $E''_i = E_i + E_{i-2} + E_{i-4} + \cdots$.

Corollary A.46. *The following determinants are equal*:

(i) $|H'_{\lambda_i-i+1} \quad H'_{\lambda_i-i+2} + H'_{\lambda_i-i} \quad \ldots \quad H'_{\lambda_i-i+k} + H'_{\lambda_i-i-k+2}|$,

(ii) $|E_{\mu_i-i+1} \quad E_{\mu_i-i+2} + E_{\mu_i-i} \quad \ldots \quad E_{\mu_i-i+l} + E_{\mu_i-i-l+2}|$,

(iii) $$|E''_{\mu_i-i+j} - E''_{\mu_i-i-j}|,$$

(iv) $$|H_{\lambda_i-i+j} - H_{\lambda_i-i-j}|.$$

Define $S_{[\lambda]}$ to be the determinant of this corollary:

$$S_{[\lambda]} = |H'_{\lambda_i-i+1} \quad H'_{\lambda_i-i+2} + H'_{\lambda_i-i} \quad \cdots \quad H'_{\lambda_i-i+k} + H'_{\lambda_i-i-k+2}|. \tag{A.47}$$

Exercise A.48*. Let Λ be the ring of symmetric polynomials, $\vartheta\colon \Lambda \to \Lambda$ the involution of Exercise A.32. Show that

$$\vartheta(S_{\langle\lambda\rangle}) = S_{[\mu]}$$

when λ and μ are conjugate partitions.

For applications to symplectic and orthogonal characters we need to specialize the variables $x_1, \ldots, x_k$. First (for the symplectic group Sp_{2n}) take $k = 2n$, let $z_1, \ldots, z_n$ be independent variables, and specialize

$$x_1 \mapsto z_1, \ \ldots, \ x_n \mapsto z_n, \ x_{n+1} \mapsto z_1^{-1}, \ \ldots, \ x_{2n} \mapsto z_n^{-1}.$$

Set

$$J_j = H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}) \tag{A.49}$$

in the field $\mathbb{Q}(z_1, \ldots, z_n)$ of rational functions.

Proposition A.50. *Given integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, we have*

$$\frac{|z_j^{\lambda_i+n-i+1} - z_j^{-(\lambda_i+n-i+1)}|}{|z_j^{n-i+1} - z_j^{-(n-i+1)}|} = |J_\lambda|,$$

where J_λ denotes the $n \times n$ matrix whose ith row is

$$(J_{\lambda_i-i+1} \quad J_{\lambda_i-i+2} + J_{\lambda_i-i} \quad \cdots \quad J_{\lambda_i-i+n} + J_{\lambda_i-i-n+2}).$$

From Proposition A.44 we obtain three other formulas for the right-hand side, e.g.,

$$|J_\lambda| = |e_{\mu_i-i+j} - e_{\mu_i-i-j}|, \tag{A.51}$$

where $e_j = E_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})$, and μ is the conjugate partition to λ.

Exercise A.52. Calculate the denominator of the left-hand side:

$$|z_j^{n-i+1} - z_j^{-(n-i+1)}| = \Delta(\xi_1, \ldots, \xi_n) \cdot \zeta_1 \cdot \ldots \cdot \zeta_n,$$

where $\xi_j = z_j + z_j^{-1}$ and $\zeta_j = z_j - z_j^{-1}$.

Proof of Proposition A.50. Set

$$\zeta_j(p) = z_j^p - z_j^{-p}, \qquad \xi_j(p) = z_j^p + z_j^{-p}. \tag{A.53}$$

By the same argument that proved the Jacobi–Trudy formula (A.5) via (A.38), the proposition follows from the following lemma:

Lemma A.54. *For $1 \le j \le n$ and any integer $l \ge 0$, $\zeta_j(l)$ is the product of the $1 \times n$, $n \times n$, and $n \times 1$ matrices*

$$(J_{l-n} \quad J_{l-n+1} + J_{l-n-1} \quad \cdots \quad J_{l-1} + J_{l-2n+1}) \cdot ((-1)^{q-p} e_{q-p}) \cdot \begin{pmatrix} \zeta_j(n) \\ \zeta_j(n-1) \\ \vdots \\ \vdots \\ \zeta_j(1) \end{pmatrix}$$

PROOF. From (A.37) we can calculate z_j^l and z_j^{-l}, and subtracting gives

$$\zeta_j(l) = \sum_{p=1}^{2n} J_{l-2n+p} s_p, \tag{A.55}$$

where $s_p = \sum_{q=p}^{2n} (-1)^{q-p} e_{q-p} \zeta_j(2n-q)$. Multiplying (A.33) by z_j^{-p} and subtracting we find

$$\begin{aligned} &\zeta_j(p) - e_1 \zeta_j(p-1) + \cdots + (-1)^{p-1} e_p \zeta_j(1) \\ &\quad = (-1)^{p+1} e_{p+1} \zeta_j(1) + (-1)^{p+2} e_{p+2} \zeta_j(2) + \cdots + e_{2n} \zeta_j(2n-p). \end{aligned} \tag{A.56}$$

Note also that

$$(-1)^p e_p = (-1)^{2n-p} e_{2n-p}, \tag{A.57}$$

since $\sum (-1)^p e_p t^p = \prod (1 - z_i t)(1 - z_i^{-1} t) = \prod (1 - \xi_i t + t^2)$. From (A.56) and (A.57) follows

$$s_{2n-p} = s_p = r_{n-p+1}, \tag{A.58}$$

where $r_p = \sum_{q=p}^{n} (-1)^{q-p} e_{q-p} \zeta_j(n+1-q)$. Combining (A.55) and (A.58) concludes the proof. □

Next (for the odd orthogonal groups O_{2n+1}) let $k = 2n+1$, and specialize the variables $x_1, \ldots, x_{2n}$ as above, and $x_{2n+1} \mapsto 1$. We introduce variables $z_j^{1/2}$ and $z_j^{-1/2}$, square roots of the variables just considered, and we work in the field $\mathbb{Q}(z_1^{1/2}, \ldots, z_n^{1/2})$. Set

$$\begin{aligned} K_j &= H_j'(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1) \\ &= H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1) - H_{j-2}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1), \end{aligned} \tag{A.59}$$

where H_j is the jth complete symmetric polynomial in $2n+1$ variables.

Proposition A.60. *Given integers $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$, we have*

$$\frac{|z_j^{\lambda_i+n-i+1/2} - z_j^{-(\lambda_i+n-i+1/2)}|}{|z_j^{n-i+1/2} - z_j^{-(n-i+1/2)}|} = |K_\lambda|,$$

where K_λ is the $n \times n$ matrix whose ith *row is*

$$(K_{\lambda_i-i+1} \quad K_{\lambda_i-i+2}+K_{\lambda_i-i} \quad \cdots \quad K_{\lambda_i-i+n}+K_{\lambda_i-i-n+2}).$$

Corollary A.46 gives three alternative expressions for this determinant, e.g.,

$$|K_\lambda| = |h_{\lambda_i-i+j} - h_{\lambda_i-i-j}|, \tag{A.61}$$

where $h_j = H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}, 1)$.

Exercise A.62. Calculate the denominator of the left-hand side:

$$|z_j^{n-i+1/2} - z_j^{-(n-i+1/2)}| = \Delta(\xi_1, \ldots, \xi_n)\cdot\zeta_1(\tfrac{1}{2})\cdot\ldots\cdot\zeta_n(\tfrac{1}{2}).$$

Proof of Proposition A.60. We have $\zeta_j(l) = z_j^l - z_j^{-l}$ and $\xi_j(l) = z_j^l + z_j^{-l}$ in $\mathbb{Q}(z_1^{1/2}, \ldots, z_n^{1/2})$ for l an integer or a half integer. First note that

$$\xi_j(\tfrac{1}{2})\cdot\zeta_j(l) = \zeta_j(l+\tfrac{1}{2}) + \zeta_j(l-\tfrac{1}{2}).$$

Multiplying the numerator and denominator of the left-hand side of the statement of the proposition by $\xi_1(\frac{1}{2})\cdot\ldots\cdot\xi_n(\frac{1}{2})$, the numerator becomes $|\zeta_j(\lambda_i+n-i+1)+\zeta_j(\lambda_i+n-i)|$, and the denominator becomes $|\zeta_j(n-i+1)+\zeta_j(n-i)| = |\zeta_j(n-i+1)|$. We can, therefore, apply Lemma A.54 to calculate the ratio, getting the determinant of a matrix whose entries are sums of certain J_j's. Note that by direct calculation $K_j = J_j + J_{j-1}$, so the terms can be combined, and the ratio is the determinant of the displayed matrix K_λ. □

Finally (for the even orthogonal groups O_{2n}), let $k = 2n$, and specialize the variables $x_1, \ldots, x_{2n}$ as above. Set

$$\begin{aligned} L_j &= H_j'(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}) \\ &= H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}) - H_{j-2}(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}), \end{aligned} \tag{A.63}$$

with H_j the complete symmetric polynomial in $2n$ variables.

Proposition A.64. *Given integers $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, we have*

$$\frac{|z_j^{\lambda_i+n-i} + z_j^{-(\lambda_i+n-i)}|}{|z_j^{n-i}+z_j^{-(n-i)}|} = \begin{cases} \frac{1}{2}|L_\lambda| & \text{if } \lambda_n > 0 \\ |L_\lambda| & \text{if } \lambda_n = 0, \end{cases}$$

where L_λ is the $n \times n$ matrix whose ith *row is*

$$(L_{\lambda_i-i+1} \quad L_{\lambda_i-i+2}+L_{\lambda_i-i} \quad \cdots \quad L_{\lambda_i-i+n}+L_{\lambda_i-i-n+2}).$$

As before, there are other expressions for these determinants, e.g.,

$$|L_\lambda| = |h_{\lambda_i-i+j} - h_{\lambda_i-i-j}|, \tag{A.65}$$

where $h_j = H_j(z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1})$.

Exercise A.66. Calculate the denominator of the left-hand side:

$$|z_j^{n-i} + z_j^{-(n-i)}| = 2 \cdot \Delta(\xi_1, \ldots, \xi_n).$$

Proof of Proposition A.64. Note that $\zeta_j \cdot \xi_j(l) = \xi_j(l+1) - \xi_j(l-1)$. Multiplying the numerator and denominator by $\zeta_1 \cdot \ldots \cdot \zeta_n$, the numerator becomes $|\zeta_j(\lambda_i + n - i + 1) - \zeta_j(\lambda_i + n - i - 1)|$ and the denominator becomes

$$|\zeta_j(n - i + 1) - \zeta_j(n - i - 1)| = 2|\zeta_j(n - i + 1)|;$$

this is seen by noting that the bottom row of the matrix on the left is $(\zeta_j(1) - \zeta_j(-1)) = (2\zeta_j(1))$, and performing row reductions starting from the bottom row. The rest of the proof is the same as in the preceding proposition. The only change is when $\lambda_n = 0$, in which case the bottom row in the numerator matrix is the same as that in the denominator. □

Exercise A.67*. Find a similar formula for

$$\frac{|z_j^{\lambda_i+n-i} - z_j^{-(\lambda_i+n-i)}|}{|z_j^{n-i} + z_j^{-(n-i)}|}.$$

APPENDIX B

On Multilinear Algebra

In this appendix we state the basic facts about tensor products and exterior and symmetric powers that are used in the text. It is hoped that a reader with some linear algebra background can fill in details of the proofs.

§B.1. Tensor Products

The *tensor product* of two vector spaces V and W over a field is a vector space $V \otimes W$ equipped with a bilinear map

$$V \times W \to V \otimes W, \qquad v \times w \mapsto v \otimes w,$$

which is universal: for any bilinear map $\beta: V \times W \to U$ to a vector space U, there is a unique linear map from $V \otimes W$ to U that takes $v \otimes w$ to $\beta(v, w)$. This universal property determines the tensor product up to canonical isomorphism. If the ground field K needs to be mentioned, the tensor product is denoted $V \otimes_K W$.

If $\{e_i\}$ and $\{f_j\}$ are bases for V and W, the elements $\{e_i \otimes f_j\}$ form a basis for $V \otimes W$. This can be used to construct $V \otimes W$. The construction is functorial: linear maps $V \to V'$ and $W \to W'$ determine a linear map from $V \otimes W$ to $V' \otimes W'$.

Similarly one has the tensor product $V_1 \otimes \cdots \otimes V_n$ of n vector spaces, with its universal multilinear map

$$V_1 \times \cdots \times V_n \to V_1 \otimes \cdots \otimes V_n,$$

taking $v_1 \times \cdots \times v_n$ to $v_1 \otimes \cdots \otimes v_n$. (Recall that a map from the Cartesian product to a vector space U is *multilinear* if, when all but one of the factors V_i are fixed, the resulting map from V_i to U is linear.) The construction of tensor products is commutative:

$$V \otimes W \cong W \otimes V, \qquad v \otimes w \mapsto w \otimes v;$$

distributive:

$$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W);$$

and associative:

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W) \cong U \otimes V \otimes W,$$

by $(u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) \mapsto u \otimes v \otimes w$.

In particular, there are *tensor powers* $V^{\otimes n} = V \otimes \cdots \otimes V$ of a fixed space V. By convention, $V^{\otimes 0}$ is the ground field.

If A is an algebra over the ground field, and V is a right A-module, and W a left A-module, there is a tensor product denoted $V \otimes_A W$, which can be constructed as the quotient of $V \otimes W$ by the subspace generated by all $(v \cdot a) \otimes w - v \otimes (a \cdot w)$ for all $v \in V$, $w \in W$, and $a \in A$. The resulting map from $V \times W$ to $V \otimes_A W$ is universal for bilinear maps β from $V \times W$ to vector spaces U that satisfy the property that $\beta(v \cdot a, w) = \beta(v, a \cdot w)$. This tensor product is also distributive.

§B.2. Exterior and Symmetric Powers

The *exterior powers* $\bigwedge^n V$ of a vector space V, sometimes denoted $\mathrm{Alt}^n V$, come equipped with an alternating multilinear map

$$V \times \cdots \times V \to \textstyle\bigwedge^n V, \qquad v_1 \times \cdots \times v_n \mapsto v_1 \wedge \cdots \wedge v_n,$$

that is universal: for β: $V \times \cdots \times V \to U$ an alternating multilinear map, there is a unique linear map from $\bigwedge^n V$ to U which takes $v_1 \wedge \cdots \wedge v_n$ to $\beta(v_1, \ldots, v_n)$. Recall that a multilinear map β is *alternating* if $\beta(v_1, \ldots, v_n) = 0$ whenever two of the vectors v_i are equal. This implies that $\beta(v_1, \ldots, v_n)$ changes sign when two of the vectors are interchanged.[1] It follows that

$$\beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \mathrm{sgn}(\sigma)\beta(v_1, \ldots, v_n) \quad \text{for all } \sigma \in \mathfrak{S}_n.$$

The exterior power can be constructed as the quotient space of $V^{\otimes n}$ by the subspace generated by all $v_1 \otimes \cdots \otimes v_n$ with two of the vectors equal. We let

$$\pi\colon V^{\otimes n} \to \textstyle\bigwedge^n V, \qquad \pi(v_1 \otimes \cdots \otimes v_n) = v_1 \wedge \cdots \wedge v_n$$

[1] This follows from the standard polarization.: for two factors, $\beta(v + w, v + w) - \beta(v, v) - \beta(w, w) = \beta(v, w) + \beta(w, v)$.

denote the projection. If $\{e_i\}$ is a basis for V, then

$$\{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_n} : i_1 < i_2 < \cdots < i_n\}$$

is a basis for $\bigwedge^n V$. Define $\bigwedge^0 V$ to be the ground field.

If V and W are vector spaces, there is a canonical linear map from $\bigwedge^a V \otimes \bigwedge^b W$ to $\bigwedge^{a+b}(V \oplus W)$, which takes $(v_1 \wedge \cdots \wedge v_a) \otimes (w_1 \wedge \cdots \wedge w_b)$ to $v_1 \wedge \cdots \wedge v_a \wedge w_1 \wedge \cdots \wedge w_b$. This determines an isomorphism

$$\bigwedge^n(V \oplus W) \cong \bigoplus_{a=0}^{n} \bigwedge^a V \otimes \bigwedge^{n-a} W. \tag{B.1}$$

(From this isomorphism the assertion about bases of $\bigwedge^n V$ follows by induction on the dimension.)

The *symmetric powers* $\mathrm{Sym}^n V$, sometimes denoted $S^n V$, comes with a universal symmetric multilinear map

$$V \times \cdots \times V \to \mathrm{Sym}^n V, \qquad v_1 \times \cdots \times v_n \mapsto v_1 \cdot \ldots \cdot v_n.$$

Recall that a multilinear map $\beta: V \times \cdots \times V \to U$ is *symmetric* if it is unchanged when any two factors are interchanged, or

$$\beta(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) = \beta(v_1, \ldots, v_n) \quad \text{for all } \sigma \in \mathfrak{S}_n.$$

The symmetric power can be constructed as the quotient space of $V^{\otimes n}$ by the subspace generated by all $v_1 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$, or by those in which σ permutes two successive factors. Again we let

$$\pi: V^{\otimes n} \to \mathrm{Sym}^n V, \qquad \pi(v_1 \otimes \cdots \otimes v_n) = v_1 \cdot \ldots \cdot v_n,$$

denote the projection. If $\{e_i\}$ is a basis for V, then

$$\{e_{i_1} \cdot e_{i_2} \cdot \ldots \cdot e_{i_n} : i_1 \le i_2 \le \cdots \le i_n\}$$

is a basis for $\mathrm{Sym}^n V$. So $\mathrm{Sym}^n V$ can be regarded as the space of homogeneous polynomials of degree n in the variables e_i. Define $\mathrm{Sym}^0 V$ to be the ground field. As before, there are canonical isomorphisms

$$\mathrm{Sym}^n(V \oplus W) \cong \bigoplus_{a=0}^{n} \mathrm{Sym}^a V \otimes \mathrm{Sym}^{n-a} W. \tag{B.2}$$

The exterior powers $\bigwedge^n V$ and symmetric powers $\mathrm{Sym}^n V$ can also be realized as subspaces of $V^{\otimes n}$, assuming, as we have throughout, that the ground field has characteristic 0. We will denote the inclusions by ι, so we have

$$V^{\otimes n} \overset{\pi}{\twoheadrightarrow} \bigwedge^n V \overset{\iota}{\rightarrowtail} V^{\otimes n}, \qquad V^{\otimes n} \overset{\pi}{\twoheadrightarrow} \mathrm{Sym}^n V \overset{\iota}{\rightarrowtail} V^{\otimes n}.$$

The imbedding $\iota: \bigwedge^n V \to V^{\otimes n}$ is defined by

$$\iota(v_1 \wedge \cdots \wedge v_n) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \tag{B.3}$$

(This is well defined since the right-hand side is alternating.) The image of ι is the space of anti-invariants of the right action of $\mathfrak{S}_n$ on $V^{\otimes n}$:

$$(v_1 \otimes \cdots \otimes v_n)\cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}, \quad v_i \in V, \sigma \in \mathfrak{S}_n. \tag{B.4}$$

(The anti-invariants are the vectors $z \in V^{\otimes n}$ such that $z \cdot \sigma = \operatorname{sgn}(\sigma) z$ for all $\sigma \in \mathfrak{S}_n$.) Moreover, if $A = \iota \circ \pi$, then $(1/n!)A$ is the projection onto this anti-invariant subspace.[2] (Often the coefficient $1/n!$ is put in front of the formula for ι; this makes no essential difference, but leads to awkward formulas for contractions.)

Similarly we have ι: $\operatorname{Sym}^n V \to V^{\otimes n}$ by

$$\iota(v_1 \cdot \ldots \cdot v_n) = \sum_{\sigma \in \mathfrak{S}_n} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}. \tag{B.5}$$

The image of ι is the space of invariants of the right action of $\mathfrak{S}_n$ on $V^{\otimes n}$. If $A = \iota \circ \pi$, then $(1/n!)A$ is the projection onto this invariant subspace.

The wedge product $\wedge$ determines a product

$$\textstyle\bigwedge^m V \otimes \bigwedge^n V \xrightarrow{\wedge} \bigwedge^{m+n} V, \tag{B.6}$$

$$(v_1 \wedge \cdots \wedge v_m) \otimes (v_{m+1} \wedge \cdots \wedge v_{m+n}) \mapsto v_1 \wedge \cdots \wedge v_m \wedge v_{m+1} \wedge \cdots \wedge v_{m+n},$$

which is associative and skew-commutative. This product is compatible with the projection from the tensor powers onto the exterior powers, but care must be taken for the inclusion of exterior in tensor powers, since for example $v \wedge w$ is sent to $v \otimes w - w \otimes v$ [not to $\frac{1}{2}(v \otimes w - w \otimes v)$] by ι. In general, the diagram

$$\begin{array}{ccc} \bigwedge^m V \otimes \bigwedge^n V & \xrightarrow{\ \wedge\ } & \bigwedge^{m+n} V \\ {\scriptstyle \iota\otimes\iota}\big\downarrow & & \big\downarrow{\scriptstyle \iota} \\ V^{\otimes m} \otimes V^{\otimes n} & \longrightarrow & V^{\otimes(m+n)} \end{array} \tag{B.7}$$

commutes when the bottom horizontal map is defined by the formula

$$\begin{aligned} &(v_1 \otimes \cdots \otimes v_m) \otimes (v_{m+1} \otimes \cdots \otimes v_{m+n}) \\ &\quad \mapsto \sum \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \otimes v_{\sigma(m+1)} \otimes \cdots \otimes v_{\sigma(m+n)}, \end{aligned} \tag{B.8}$$

the sum over all "shuffles," i.e., permutations σ of $\{1, \ldots, m+n\}$ that preserve the order of the subsets $\{1, \ldots, m\}$ and $\{m+1, \ldots, m+n\}$.

Similarly the symmetric powers have a commutative product $(v_1 \cdot \ldots \cdot v_m) \otimes (v_{m+1} \cdot \ldots \cdot v_{m+n}) \mapsto v_1 \cdot \ldots \cdot v_m \cdot v_{m+1} \cdot \ldots \cdot v_{m+n}$, with a similar compatibility. Note that $v^2 \in \operatorname{Sym}^2 V$ is sent to $2v \otimes v$ in $V \otimes V$, $v^n \in \operatorname{Sym}^n V$ to $n!(v \otimes \cdots \otimes v)$ in $V^{\otimes n}$, and generally one has the analogue of (B.7), changing each "sgn(σ)" to "1" in formula (B.8).

All these mappings are compatible with linear maps of vector spaces $V \to W$, and in particular commute with the left actions of the general linear group $GL(V) = \operatorname{Aut}(V)$ of automorphisms, or the algebra $\operatorname{End}(V) = \operatorname{Hom}(V, V)$ of endomorphisms, on $V^{\otimes n}$, $\bigwedge^n V$, and $\operatorname{Sym}^n V$.

[2] It is this factor which limits our present discussion to vector spaces over fields of characteristic 0.

It is sometimes convenient to make algebras out of the direct sum of all of the tensor, exterior, or symmetric powers. The *tensor algebra* $T^{\cdot}V$ is the sum $\bigoplus_{n\geq 0} V^{\otimes n}$, with product determined by the canonical isomorphism $V^{\otimes n} \otimes V^{\otimes m} \to V^{\otimes(n+m)}$. The *exterior algebra* $\bigwedge^{\cdot}V$ is the sum $\bigoplus_{n\geq 0} \bigwedge^n V$, which is the quotient of $T^{\cdot}V$ by the two-sided ideal generated by all $v \otimes v$ in $V^{\otimes 2}$. The *symmetric algebra* $\mathrm{Sym}^{\cdot}V$ is the sum $\bigoplus_{n\geq 0} \mathrm{Sym}^n V$, which is the quotient of $T^{\cdot}V$ by the two-sided ideal generated by all $v \otimes w - w \otimes v$ in $V^{\otimes 2}$.

Exercise B.9. The algebra $\mathrm{Sym}^{\cdot}V$ is a commutative, graded algebra, which satisfies the universal property that any linear map from V to the first graded piece C^1 of a commutative graded algebra $C^{\cdot}$ determines a homomorphism $\mathrm{Sym}^{\cdot}V \to C^{\cdot}$ of graded algebras. Use this to show that $\mathrm{Sym}^{\cdot}(V \oplus W) \cong \mathrm{Sym}^{\cdot}V \otimes \mathrm{Sym}^{\cdot}W$, and deduce the isomorphism (B.2). Prove the analogous assertions for $\bigwedge^{\cdot}V$, in the category of skew-commutative graded algebras. In particular, construct an isomorphism $\bigwedge^{\cdot}(V \oplus W) \cong \bigwedge^{\cdot}V \mathbin{\hat{\otimes}} \bigwedge^{\cdot}W$, where $\hat{\otimes}$ denotes the skew-commutative tensor product: it is the usual tensor product additively, but the product has $(a \otimes b)\cdot(c \otimes d) = (-1)^{\deg(b)\deg(c)}(a \cdot b) \otimes (c \cdot d)$ for homogeneous elements a and c in the first algebra, and b and d in the second. In particular, this proves (B.1).

§B.3. Duals and Contractions

Although only a few simple contractions are used in the lectures, and most of these are written out by hand where needed, it may be useful to see the general picture.

If V^* denotes the dual space to V, there are contraction maps

$$c_j^i\colon V^{\otimes p} \otimes (V^*)^{\otimes q} \to V^{\otimes(p-1)} \otimes (V^*)^{\otimes(q-1)},$$

for any $1 \leq i \leq p$ and $1 \leq j \leq q$, determined by evaluating the jth coordinate of $(V^*)^{\otimes q}$ on the ith coordinate of $V^{\otimes p}$:

$$\begin{aligned} &c_j^i(v_1 \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \varphi_q) \\ &\quad = \varphi_j(v_i)v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \hat{\varphi}_j \otimes \cdots \otimes \varphi_q. \end{aligned} \tag{B.10}$$

More generally if $I = (i_1, \ldots, i_n)$ and $J = (j_1, \ldots, j_n)$ are two sequences of n distinct indices from $\{1, \ldots, p\}$ and $\{1, \ldots, q\}$, respectively, there is a contraction

$$c_J^I\colon V^{\otimes p} \otimes (V^*)^{\otimes q} \to V^{\otimes(p-n)} \otimes (V^*)^{\otimes(q-n)} \tag{B.11}$$

which takes $v_1 \wedge \cdots \wedge v_p \otimes \varphi_1 \otimes \cdots \otimes \varphi_q$ to

$$\prod_{\alpha=1}^{n} \varphi_{j_\alpha}(v_{i_\alpha})v_1 \otimes \cdots \otimes \hat{v}_{i_1} \otimes \cdots \otimes \hat{v}_{i_2} \otimes \cdots \otimes v_p \otimes \varphi_1 \otimes \cdots \otimes \hat{\varphi}_{j_1} \otimes \cdots \otimes \varphi_q.$$

For example, if $p = q = n$ and $I = J = (1, \ldots, n)$, this contraction $V^{\otimes n} \otimes (V^*)^{\otimes n} \to \mathbb{C}$ identifies $(V^*)^{\otimes n}$ with the dual space of $V^{\otimes n}$.

Now $(V^{\otimes n})^*$ consists of n-multilinear forms on V, and $(\bigwedge^n V)^*$ consists of alternating n multilinear forms on V; in particular, $(\bigwedge^n V)^*$ is a subspace of $(V^{\otimes n})^*$; this is the inclusion via π^*. The composite

$$\bigwedge^n(V^*) \to (V^*)^{\otimes n} \to (V^{\otimes n})^*,$$

where the first map is the inclusion ι and the second is the isomorphism of the preceding paragraph, maps $\bigwedge^n(V^*)$ isomorphically onto the subspace $(\bigwedge^n V)^*$. Explicitly,

$$\bigwedge^n(V^*) \xrightarrow{\cong} (\bigwedge^n V)^*,$$

$$\varphi_1 \wedge \cdots \wedge \varphi_n \mapsto [v_1 \wedge \cdots \wedge v_n \mapsto \sum \operatorname{sgn}(\sigma)\varphi_{\sigma(1)}(v_1)\cdot \ldots \cdot \varphi_{\sigma(n)}(v_n)$$
$$= \det(\varphi_j(v_i))].$$

This dual pairing $\bigwedge^n V \otimes \bigwedge^n(V^*) \to K$ is often denoted $\langle\ ,\ \rangle$.

There is a similar isomorphism of $\operatorname{Sym}^n(V^*)$ with $\operatorname{Sym}^n(V)^*$, but without the signs "$\operatorname{sgn}(\sigma)$."

Exercise B.12. If $e_1, \ldots, e_m$ is a basis for V, with e_i^* the dual basis for V^*, then $\{e_{i_1} \wedge \cdots \wedge e_{i_n}: 1 \le i_1 < \cdots < i_n \le m\}$ is a basis for $\bigwedge^n V$, and $\{e_1^{i_1} \cdot \ldots \cdot e_m^{i_m}: i_\alpha \ge 0, \sum i_\alpha = n\}$ is a basis for $\operatorname{Sym}^n V$. Show that, via the above isomorphisms, the dual bases for $\bigwedge^n(V^*)$ and $\operatorname{Sym}^n(V^*)$ are

$$\{e_{i_1}^* \wedge \cdots \wedge e_{i_n}^*\} \quad \text{and} \quad \left\{\frac{1}{\prod_\alpha (i_\alpha!)}(e_1^*)^{i_1}\cdot \ldots \cdot (e_m^*)^{i_m}\right\}.$$

There are related contractions, sometimes called internal products, and denoted $\lrcorner$ and $\llcorner$, on exterior and symmetric powers. For the exterior powers they are maps:

$$\begin{aligned} &\textstyle\bigwedge^p V \otimes \bigwedge^{p+q}(V^*) \to \bigwedge^q(V^*), && x \otimes \alpha \mapsto x \lrcorner \alpha; \\ &\textstyle\bigwedge^{p+q} V \otimes \bigwedge^p(V^*) \to \bigwedge^q(V), && x \otimes \alpha \mapsto x \llcorner \alpha. \end{aligned} \tag{B.13}$$

These can be defined most simply as transposes of wedge products, i.e., they are determined by the identities

$$\langle z, x \lrcorner \alpha\rangle = \langle z \wedge x, \alpha\rangle \quad \text{for } z \in \textstyle\bigwedge^q V$$

and

$$\langle x \llcorner \alpha, \beta\rangle = \langle x, \alpha \wedge \beta\rangle \quad \text{for } \beta \in \textstyle\bigwedge^q(V^*).$$

(The relation of this definition to the contraction maps c_J^I above is expressed in Exercise B.16.) Note that when $q = 0$, these contractions reduce to the previous duality pairing between $\bigwedge^p V$ and $\bigwedge^p(V^*)$.

For symmetric powers, the internal products are defined similarly:

$$\begin{aligned} &\operatorname{Sym}^p V \otimes \operatorname{Sym}^{p+q}(V^*) \to \operatorname{Sym}^q(V^*), && x \otimes \alpha \mapsto x \lrcorner \alpha; \\ &\operatorname{Sym}^{p+q} V \otimes \operatorname{Sym}^p(V^*) \to \operatorname{Sym}^q(V), && x \otimes \alpha \mapsto x \llcorner \alpha. \end{aligned} \tag{B.14}$$

Exercise B.15. For $v, w \in V$, and $\varphi, \psi \in V^*$, show that

$$v \lrcorner (\varphi \wedge \psi) = \psi(v)\varphi - \varphi(v)\psi \quad \text{and} \quad (v \wedge w) \llcorner \varphi = \varphi(v)w - \varphi(w)v.$$

More generally, for if $x = v_1 \wedge \cdots \wedge v_p$ and $\alpha = \varphi_1 \wedge \cdots \wedge \varphi_{p+q}$, with $v_i \in V$ and $\varphi_j \in V^*$, then

(i) $$x \lrcorner \alpha = \sum \operatorname{sgn}(\sigma)\varphi_{\sigma(q+1)}(v_1) \cdot \ldots \cdot \varphi_{\sigma(q+p)}(v_p) \cdot \varphi_{\sigma(1)} \wedge \cdots \wedge \varphi_{\sigma(q)},$$

the sum over all permutations σ of $\{1, \ldots, p+q\}$ that preserve the order of $\{1, \ldots, q\}$. If $x = v_1 \wedge \cdots \wedge v_{p+q}$ and $\alpha = \varphi_1 \wedge \cdots \wedge \varphi_p$, then

(ii) $$x \llcorner \alpha = \sum \operatorname{sgn}(\sigma)\varphi_1(v_{\sigma(1)}) \cdot \ldots \cdot \varphi_p(v_{\sigma(p)}) \cdot v_{\sigma(p+1)} \wedge \cdots \wedge v_{\sigma(p+q)},$$

the sum over all permutations that preserve the order of $\{p+1, \ldots, p+q\}$. Verify these formulas and use them to give formulas for these internal products in terms of standard bases. State and verify analogous formulas for symmetric powers. In particular, for $v, w \in V$, $\varphi, \psi \in V^*$,

$$v \lrcorner (\varphi \cdot \psi) = \psi(v)\varphi + \varphi(v)\psi \quad \text{and} \quad (v \cdot w) \llcorner \varphi = \varphi(v)w + \varphi(w)v.$$

For example, $v \lrcorner (\varphi^2) = 2\varphi(v)\varphi$ and $(v^2) \llcorner \varphi = 2\varphi(v)v$.

Exercise B.16. Using formula (ii) of the preceding exercise, show that the contraction map $\llcorner$ may be given as $1/p!q!$ times the composition of the maps

$$\textstyle\bigwedge^{p+q} V \otimes \bigwedge^p(V^*) \to V^{\otimes(p+q)} \otimes (V^*)^{\otimes p} \to V^{\otimes q} \to \bigwedge^q V,$$

where the middle map is the contraction map c_J^I of (B.11), with $I = J = \{1, \ldots, p\}$, and the other maps come from ι and π. Prove the same formulas (with the same scalar factor) for the other internal products.

Exercise B.17. In the situation of formula (ii), suppose the v_i are independent, and let W be the $(p+q)$-dimensional subspace of V that they span; suppose the φ_i are independent, and let Z be the p-codimensional subspace of V of the common zeros of the φ_i. Show that $x \llcorner \alpha = 0$ if $\dim(W \cap Z) > q$, and otherwise $x \llcorner \alpha = u_1 \wedge \cdots \wedge u_q$ for some vectors u_i that span $W \cap Z$.

Exercise B.18. Prove the formulas

$$(x \wedge y) \lrcorner \alpha = x \lrcorner (y \lrcorner \alpha) \quad \text{and} \quad x \llcorner (\alpha \wedge \beta) = (x \llcorner \alpha) \llcorner \beta.$$

State and verify the analogous formulas for symmetric powers.

For a detailed development of these ideas, see [Bour, *Algebra*, Chap. 3].

APPENDIX C

On Semisimplicity

§C.1. The Killing Form and Cartan's Criterion

We recall first the Jordan decomposition of a linear transformation X of a finite-dimensional complex vector space V as a sum of its semisimple and nilpotent parts: $X = X_s + X_n$, where X_s is the semisimple part of X, and X_n the nilpotent part. It is uniquely characterized by the fact that X_s is semisimple (diagonalizable), X_n is nilpotent, and X_s and X_n commute with each other. In fact, X_s and X_n can be written as polynomials in X, so any endomorphism that commutes with X automatically commutes with X_s and X_n. One case of the invariance of Jordan decomposition is an easy calculation:

Exercise C.1*. For any $X \in \mathfrak{gl}(V)$, the endomorphism $\mathrm{ad}(X)$ of $\mathfrak{gl}(V)$ satisfies

$$\mathrm{ad}(X)_s = \mathrm{ad}(X_s) \quad \text{and} \quad \mathrm{ad}(X)_n = \mathrm{ad}(X_n).$$

There is a Killing form B_V defined on $\mathfrak{gl}(V)$ by the formula

$$B_V(X, Y) = \mathrm{Tr}(X \circ Y), \tag{C.2}$$

where Tr is the trace and $\circ$ denotes composition of transformations. As in (14.23), the identity

$$B_V(X, [Y, Z]) = B_V([X, Y], Z) \tag{C.3}$$

holds for all X, Y, Z in $\mathfrak{gl}(V)$.

The Killing form B on a Lie algebra $\mathfrak{g}$ is that of Exercise C.1 for the adjoint representation: $B(X, Y) = B_{\mathfrak{g}}(\mathrm{ad}(X), \mathrm{ad}(Y))$. This was introduced in Lecture 14, where a few of its properties were proved. Here we use the Killing form to characterize solvability and semisimplicity of the Lie algebra.

If $\mathfrak{g}$ is solvable, by Lie's theorem its adjoint representation can be put in upper-triangular form. It follows that $\mathscr{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ acts by strictly upper-triangular matrices. So if X is in $\mathscr{D}\mathfrak{g}$ and Y in $\mathfrak{g}$, then $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ is strictly upper triangular; in particular its trace $B(X, Y)$ is zero. Cartan's criterion is that this characterizes solvability:

Proposition C.4. *The Lie algebra $\mathfrak{g}$ is solvable if and only if $B(\mathfrak{g}, \mathscr{D}\mathfrak{g}) = 0$.*

We will prove something that looks a little weaker, but will turn out to be a little stronger. We prove:

Theorem C.5 (Cartan's criterion). *If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$ and $B_V(X, Y) = 0$ for all X and Y in $\mathfrak{g}$, then $\mathfrak{g}$ is solvable.*

For this, it suffices to show that every element of $\mathscr{D}\mathfrak{g}$ is nilpotent, for then by Engel's theorem $\mathscr{D}\mathfrak{g}$ must be a nilpotent ideal, and therefore $\mathfrak{g}$ is solvable.

So take $X \in \mathscr{D}\mathfrak{g}$, and let $\lambda_1, \ldots, \lambda_r$ be its eigenvalues (counted with multiplicity) for X as an endomorphism of V. We must show the λ_i are all zero. These eigenvalues satisfy some obvious relations; for example, $\sum \lambda_i \lambda_i = \mathrm{Tr}(X \circ X)) = B_V(X, X) = 0$. What we need to show is

$$\bar{\lambda}_1 \lambda_1 + \cdots + \bar{\lambda}_r \lambda_r = 0. \tag{C.6}$$

To prove this, take a basis for V so that X is in Jordan canonical form, with $\lambda_1, \ldots, \lambda_r$ down the diagonal; the semisimple part $D = X_s$ of X is this diagonal transformation. Let $\bar{D}$ be the endomorphism of V given by the diagonal matrix with $\bar{\lambda}_1, \ldots, \bar{\lambda}_r$ down the diagonal. Since $\mathrm{Tr}(\bar{D} \circ X) = \sum \bar{\lambda}_i \lambda_i$, it suffices to prove

$$\mathrm{Tr}(\bar{D} \circ X) = 0. \tag{C.7}$$

Since X is a sum of commutators $[Y, Z]$, with Y and Z in $\mathfrak{g}$, $\mathrm{Tr}(\bar{D} \circ X)$ is a sum of terms of the form $\mathrm{Tr}(\bar{D} \circ [Y, Z]) = \mathrm{Tr}([\bar{D}, Y] \circ Z)$. So we will be done if we know that $[\bar{D}, Y]$ belongs to $\mathfrak{g}$, for our hypothesis is that $\mathrm{Tr}(\mathfrak{g} \circ \mathfrak{g}) \equiv 0$. That is, we are reduced to showing

$$\mathrm{ad}(\bar{D})(\mathfrak{g}) \subset \mathfrak{g}. \tag{C.8}$$

For this it suffices to prove that $\mathrm{ad}(\bar{D})$ can be written as a polynomial in $\mathrm{ad}(X)$, for we know that $\mathrm{ad}(X)^k(Y)$ is in $\mathfrak{g}$ if X and Y are in $\mathfrak{g}$. Since $\mathrm{ad}(D) = \mathrm{ad}(X_s) = \mathrm{ad}(X)_s$ is a polynomial in $\mathrm{ad}(X)$, it suffices to show that $\mathrm{ad}(\bar{D})$ can be written as a polynomial in $\mathrm{ad}(D)$. This is a simple computation: using the usual basis $\{E_{ij}\}$ for $\mathfrak{gl}(V)$, $\mathrm{ad}(D)$ and $\mathrm{ad}(\bar{D})$ are complex conjugate diagonal matrices, and any such are polynomials in each other. □

We can prove now that if $\mathfrak{g}$ is a Lie algebra for which $B(\mathscr{D}\mathfrak{g}, \mathscr{D}\mathfrak{g}) \equiv 0$, then $\mathfrak{g}$ is solvable, which certainly implies Proposition C.4. By what we just proved, the image of $\mathscr{D}\mathfrak{g}$ by the adjoint representation in $\mathfrak{gl}(\mathfrak{g})$ is solvable. Since the kernel of the adjoint map is abelian, this makes $\mathscr{D}\mathfrak{g}$ solvable (cf. Exercise 9.8), and by definition this makes $\mathfrak{g}$ solvable. □

Exercise C.9. Show that a Lie algebra $\mathfrak{g}$ is solvable if and only if $B(\mathrm{ad}(X), \mathrm{ad}(X)) = 0$ for all X in $\mathfrak{g}$.

It is easy to deduce from Cartan's criterion a criterion for semisimplicity—part of which we saw in Lecture 14, but there assuming some facts we had not proved yet:

Proposition C.10. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form B is nondegenerate.*

Proof. By (C.3) the null-space $\mathfrak{s} = \{X \in \mathfrak{g}: B(X, Y) = 0 \text{ for all } Y \in \mathfrak{g}\}$ is an ideal. Suppose $\mathfrak{g}$ is semisimple. By Cartan's criterion, the image $\mathrm{ad}(\mathfrak{s}) \subset \mathfrak{gl}(\mathfrak{g})$ is solvable; as in the preceding proof, $\mathfrak{s}$ is then solvable, so $\mathfrak{s} = 0$ by the definition of semisimple. Conversely, if B is nondegenerate, we must show that any abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$ must be zero. If $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$, then $A = \mathrm{ad}(X) \circ \mathrm{ad}(Y)$ maps $\mathfrak{g}$ into $\mathfrak{a}$ and $\mathfrak{a}$ to 0, so $\mathrm{Tr}(A) = 0$. This implies that $\mathfrak{a} \subset \mathfrak{s} = 0$, as required. □

Corollary C.11. *A semisimple Lie algebra is a direct product of simple Lie algebras.*

Proof. For any ideal $\mathfrak{h}$ of $\mathfrak{g}$, the annihilator

$$\mathfrak{h}^{\perp} = \{X \in \mathfrak{g}: B(X, Y) = 0 \text{ for all } Y \in \mathfrak{h}\}$$

is an ideal, by (C.3) again. By Cartan's criterion, $\mathfrak{h} \cap \mathfrak{h}^{\perp}$ is solvable, hence zero, so $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$. The decomposition follows by a simple induction. □

It follows that $\mathfrak{g} = \mathscr{D}\mathfrak{g}$, and that all ideals and images of $\mathfrak{g}$ are semisimple. In fact:

Exercise C.12*. Show that if $\mathfrak{g}$ is a direct product of simple Lie algebras, the only ideals in $\mathfrak{g}$ are sums of some of the factors. In particular, the decomposition into simple factors is unique (not just up to isomorphism).

Exercise C.13*. Show that if $\mathfrak{g}$ is semisimple, the adjoint map $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is an isomorphism of $\mathfrak{g}$ onto the algebra $\mathrm{Der}(\mathfrak{g})$ of derivations of $\mathfrak{g}$.

Exercise C.14. Show that if $\mathfrak{g}$ is nilpotent then its Killing form is identically zero, and find a counterexample to the converse.

§C.2. Complete Reducibility and the Jordan Decomposition

We repeat that this section is optional, since the results can be deduced from the existence of a compact group such that the complexification of its Lie algebra is a given semisimple Lie algebra. We include here the standard algebraic approach. A finite-dimensional representation of a Lie algebra $\mathfrak{g}$ will be called a $\mathfrak{g}$-module, and a $\mathfrak{g}$-invariant subspace a submodule.

Proposition C.15. *Let V be a representation of the semisimple Lie algebra $\mathfrak{g}$ and $W \subset V$ a submodule. Then there exists a submodule $W' \subset V$ complementary to W.*

PROOF. Since the image of $\mathfrak{g}$ by the representation is semisimple, we may assume $\mathfrak{g} \subset \mathfrak{gl}(V)$. We will require a slight generalization of the Casimir operator $C_V \in \operatorname{End}(V)$ which was used in §25.1 in the proof of Freudenthal's formula. We take a basis $U_1, \ldots, U_r$ for $\mathfrak{g}$, and a dual basis $U'_1, \ldots, U'_r$, but this time with respect to the Killing form B_V defined in Exercise C.1: $B_V(X, Y) = \operatorname{Tr}(X \circ Y)$. (Note by Cartan's criterion that B_V is nondegenerate.) Then C_V is defined by the formula $C_V(v) = \sum U_i \cdot (U'_i \cdot v)$.

As before, a simple calculation shows that C_V is an endomorphism of V that commutes with the action of $\mathfrak{g}$. Its trace is

$$\operatorname{Tr}(C_V) = \sum \operatorname{Tr}(U_i \circ U'_i) = \sum B_V(U_i, U'_i) = \dim(\mathfrak{g}). \tag{C.16}$$

We note also that since C_V maps any submodule W to itself, and since it commutes with $\mathfrak{g}$, its kernel $\operatorname{Ker}(C_V)$ and image are submodules.

Note first that all one-dimensional representations of a semisimple $\mathfrak{g}$ are trivial, since $\mathscr{D}\mathfrak{g}$ must act trivially on a one-dimensional representation, and $\mathfrak{g} = \mathscr{D}\mathfrak{g}$.

We proceed to the proof itself. As should be familiar from Lecture 9, the basic case to prove is when $W \subset V$ is an irreducible invariant subspace of codimension one. Then C_V maps W into itself, and C_V acts trivially on V/W. But now by Schur's lemma, since W is irreducible, C_V is multiplication by a scalar on W. This scalar is not zero, or (C.16) would be contradicted. Hence $V = W \oplus \operatorname{Ker}(C_V)$, which finishes this special case.

It follows easily by induction on the dimension that the same is true whenever $W \subset V$ has codimension one. For if W is not irreducible, let Z be a nonzero submodule, and find a complement to $W/Z \subset V/Z$ (by induction), say Y/Z. Since Y/Z is one dimensional, find (by induction) U so that $Y = Z \oplus U$. Then $V = W \oplus U$.

By the same argument, it suffices to prove the statement of the theorem when W is irreducible. Consider the restriction map

$$\rho\colon \operatorname{Hom}(V, W) \to \operatorname{Hom}(W, W),$$

a homomorphism of g-modules. The second contains the one-dimensional submodule $\mathrm{Hom}_{\mathfrak{g}}(W, W)$. By the preceding case, there is a one-dimensional submodule of $\rho^{-1}(\mathrm{Hom}_{\mathfrak{g}}(W, W)) \subset \mathrm{Hom}(V, W)$ which maps onto $\mathrm{Hom}_{\mathfrak{g}}(W, W)$ by ρ. Since one-dimensional modules are trivial, this means there is a $\mathfrak{g}$-invariant ψ in $\mathrm{Hom}(V, W)$ such that $\rho(\psi) = 1$. But this means that ψ is a $\mathfrak{g}$-invariant projection of V onto W, so $V = W \oplus \mathrm{Ker}(\psi)$, as required. □

We will apply this to prove the invariance of Jordan decomposition (Theorem 9.20). The essential point is:

Proposition C.17. *Let $\mathfrak{g}$ be a semisimple Lie subalgebra of $\mathfrak{gl}(V)$. Then for any element $X \in \mathfrak{g}$, the semisimple part X_s and the nilpotent part X_n are also in $\mathfrak{g}$.*

Proof. The idea is to write $\mathfrak{g}$ as an intersection of Lie subalgebras of $\mathfrak{gl}(V)$ for which the conclusion of the theorem is easy to prove. For example, we know $\mathfrak{g} \subset \mathfrak{sl}(V)$ since $\mathfrak{g} = \mathscr{D}\mathfrak{g}$, and clearly X_s and X_n are traceless if X is. Similarly, if V is not irreducible, for any submodule W of V, let

$$\mathfrak{s}_W = \{Y \in \mathfrak{gl}(V): Y(W) \subset W \text{ and } \mathrm{Tr}(Y|_W) = 0\}.$$

Then $\mathfrak{g}$ is also a subalgebra of $\mathfrak{s}_W$, and X_s and X_n are also in $\mathfrak{s}_W$.

Since $[X, \mathfrak{g}] \subset \mathfrak{g}$, it follows that $[p(X), \mathfrak{g}] \subset \mathfrak{g}$ for any polynomial $p(T)$. Hence $[X_s, \mathfrak{g}] \subset \mathfrak{g}$ and $[X_n, \mathfrak{g}] \subset \mathfrak{g}$. In other words, X_s and X_n belong to the Lie subalgebra $\mathfrak{n}$ of $\mathfrak{gl}(V)$ consisting of those endomorphisms A such that $[A, \mathfrak{g}] \subset \mathfrak{g}$. So $\mathfrak{n}$ gives us another subalgebra to work with. Now we claim that $\mathfrak{g}$ is the intersection of $\mathfrak{n}$ and all the algebras $\mathfrak{s}_W$ for all submodules W of V. This claim, as we saw, will finish the proof. Let $\mathfrak{g}'$ be the intersection of all these Lie algebras. Then $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$ since $\mathfrak{g}' \subset \mathfrak{n}$.

By the complete reducibility theorem we can find a submodule U of $\mathfrak{g}'$ so that $\mathfrak{g}' = \mathfrak{g} \oplus U$. Since $[\mathfrak{g}, \mathfrak{g}'] \subset \mathfrak{g}$, we must have $[\mathfrak{g}, U] = 0$. To show that U is 0, it suffices to show that for any $Y \in U$ its restriction to any irreducible submodule W of V is zero (noting that Y preserves W since $Y \in \mathfrak{s}_W$, and that V is a sum of irreducible submodules). But since Y commutes with $\mathfrak{g}$, Schur's lemma implies that the restriction of Y to W is multiplication by a scalar, and the assumption that $Y \in \mathfrak{s}_W$ means that $\mathrm{Tr}(Y|_W) = 0$, so $Y|_W = 0$, as required. □

Now if $\mathfrak{g}$ is a semisimple algebra, the adjoint representation ad embeds $\mathfrak{g}$ in $\mathfrak{gl}(\mathfrak{g})$. For any X in $\mathfrak{g}$ the theorem implies that the semisimple and nilpotent parts of $\mathrm{ad}(X)$ are in $\mathfrak{g}$. We write these X_s and X_n. The decomposition $X = X_s + X_n$ may be called the *absolute* Jordan decomposition. Note that $[X_s, X_n] = 0$. It follows easily from the definition that if $\rho: \mathfrak{g} \to \mathfrak{g}'$ is a homomorphism from one semisimple Lie algebra onto another, then $\rho(X_s) = \rho(X)_s$ and $\rho(X_n) = \rho(X)_n$. (This follows for example from the fact that $\mathfrak{g}'$ is obtained from $\mathfrak{g}$ by factoring out some of its simple ideals.) In fact, the absolute decomposition determines all others:

Corollary C.18. *If $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$ is any representation of a semisimple Lie algebra $\mathfrak{g}$, then $\rho(X_s)$ is the semisimple part of $\rho(X)$ and $\rho(X_n)$ is the nilpotent part of $\rho(X)$.*

PROOF. We just saw that $\rho(X_s)$ and $\rho(X_n)$ are the semisimple and nilpotent parts of $\rho(X)$ as regarded in the semisimple Lie algebra $\mathfrak{g}' = \rho(\mathfrak{g})$. Apply the theorem to $\mathfrak{g}' \subset \mathfrak{gl}(V)$. □

It follows that an element X in a semisimple Lie algebra that is semisimple in one faithful representation is semisimple in all representations.

§C.3. On Derivations

In this final section we collect a few facts relating the Killing form, solvability, and nilpotency with derivations of Lie algebras, mainly for use in Appendix E. We first prove a couple of lemmas related to the Lie–Engel theory of Lecture 9. For these $\mathfrak{g}$ is any Lie algebra, $\mathfrak{r} = \mathrm{Rad}(\mathfrak{g})$ denotes its radical, and $\mathscr{D}\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Lemma C.19. *For any representation $\rho: \mathfrak{g} \to \mathfrak{gl}(V)$, every element of $\rho(\mathscr{D}\mathfrak{g} \cap \mathfrak{r})$ is a nilpotent endomorphism.*

PROOF. It suffices to treat the case where the representation V is irreducible, for if W were a proper subrepresentation, we would know the result by induction on the dimension for W and V/W, which implies it for V. We may replace $\mathfrak{g}$ by its image, so we may assume ρ is injective. In this case we show that $\mathscr{D}\mathfrak{g} \cap \mathfrak{r} = 0$. We may assume $\mathfrak{r} \neq 0$. Consider the largest integer k such that $\mathfrak{a} = \mathscr{D}^k\mathfrak{r}$ is not zero. This $\mathfrak{a}$ is an abelian ideal of $\mathfrak{g}$. It suffices to show that $\mathscr{D}\mathfrak{g} \cap \mathfrak{a} = 0$, for if $k > 0$, then $\mathfrak{a} \subset \mathscr{D}\mathfrak{g}$.

We need three facts:

(i) If $\mathfrak{g} \subset \mathfrak{gl}(V)$ is an irreducible representation and $\mathfrak{b}$ is any ideal of $\mathfrak{g}$ that consists of nilpotent transformations of V, then $\mathfrak{b} = 0$. (Indeed, by Engel's theorem,

$$W = \{v \in V: X(v) = 0 \text{ for all } X \in \mathfrak{b}\}$$

is nonzero, and by Lemma 9.13, W is preserved by $\mathfrak{g}$. Since V is irreducible, $W = V$, which says that $\mathfrak{b} = 0$.)

(ii) A transformation X is nilpotent exactly when $\mathrm{Tr}(X^n) = 0$ for all positive integers n. (This is seen by writing X in Jordan canonical form.)

(iii) $\mathrm{Tr}([X, Y] \cdot Z) = 0$ whenever $[Y, Z] = 0$. (This follows from the identity (C.3): $\mathrm{Tr}([X, Y] \cdot Z) = \mathrm{Tr}(X \cdot [Y, Z])$.)

Next we can see that $[\mathfrak{g}, \mathfrak{a}] = 0$. For if $X \in \mathfrak{g}$ and $Y \in \mathfrak{a}$, then $[X, Y] \in \mathfrak{a}$; since $\mathfrak{a}$ is abelian, Y commutes with $[X, Y]$ and hence with powers of $[X, Y]$.

Applying (iii) with $Z = [X, Y]^{n-1}$ gives $\mathrm{Tr}([X, Y]^n) = 0$ for $n > 0$, and (ii) and (i) imply that $[\mathfrak{g}, \mathfrak{a}] = 0$.

Finally we show that $\mathscr{D}\mathfrak{g} \cap \mathfrak{a} = 0$. If $X, Y \in \mathfrak{g}$ and $[X, Y] \in \mathfrak{a}$, then $[Y, [X, Y]] = 0$ by the preceding step, so again Y commutes with powers of $[X, Y]$, and the same argument shows that $\mathrm{Tr}([X, Y]^n) = 0$, and (ii) and (i) again show that $\mathscr{D}\mathfrak{g} \cap \mathfrak{a} = 0$. □

Lemma C.20. *For any Lie algebra* $\mathfrak{g}$, $[\mathfrak{g}, \mathfrak{r}]$ *is nilpotent.*

PROOF. Look at the images $\bar{\mathfrak{g}}$ and $\bar{\mathfrak{r}}$ of $\mathfrak{g}$ and $\mathfrak{r}$ by the adjoint representation ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. By Lemma C.19 and Engel's theorem, $[\bar{\mathfrak{g}}, \bar{\mathfrak{r}}]$ is a nilpotent ideal of $\bar{\mathfrak{g}}$. Since the kernel of the adjoint representation is the center of $\mathfrak{g}$, it follows that the quotient of $[\mathfrak{g}, \mathfrak{r}]$ by a central ideal is nilpotent, which implies that $[\mathfrak{g}, \mathfrak{r}]$ itself is nilpotent. □

An ideal $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is called *characteristic* if any derivation of $\mathfrak{g}$ maps $\mathfrak{a}$ into itself. Note that an ideal is just a subspace that is preserved by all inner derivations $D_X = \mathrm{ad}(X)$. It follows from the definitions that if $\mathfrak{a}$ is any ideal in $\mathfrak{g}$, then any characteristic ideal in $\mathfrak{a}$ is automatically an ideal in $\mathfrak{g}$.

The following simple construction is useful for turning questions about general derivations into questions about inner derivations. Given any Lie algebra $\mathfrak{g}$ and a derivation D of $\mathfrak{g}$, let $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}$, and define a bracket on $\mathfrak{g}'$ by

$$[(X, \lambda), (Y, \mu)] = ([X, Y] + \lambda D(Y) - \mu D(X), 0).$$

It is easy to verify that $\mathfrak{g}'$ is a Lie algebra containing $\mathfrak{g} = \mathfrak{g} \oplus 0$ as an ideal, and that, setting $\xi = (0, 1)$, the restriction of $D_\xi = \mathrm{ad}(\xi)$ to $\mathfrak{g}$ is the given derivation D.

As a simple application of this construction, if B is the Killing form on $\mathfrak{g}$, we have the identity

$$B(D(X), Y) + B(X, D(Y)) = 0 \tag{C.21}$$

for any derivation D of $\mathfrak{g}$, and any X and Y in $\mathfrak{g}$. Indeed, if B' is the Killing form on $\mathfrak{g}'$, (C.3) gives $B'([\xi, X], Y) + B'(X, [\xi, Y]) = 0$; since $\mathfrak{g}$ is an ideal in $\mathfrak{g}'$, B is the restriction of B' to $\mathfrak{g}$, and (C.21) follows.

From (C.21) it follows that if $\mathfrak{a}$ is a characteristic ideal of $\mathfrak{g}$, then its orthogonal complement with respect to the Killing form is also a characteristic ideal of $\mathfrak{g}$.

Proposition C.22. *For any Lie algebra* $\mathfrak{g}$, $\mathrm{Rad}(\mathfrak{g})$ *is the orthogonal complement to* $\mathscr{D}\mathfrak{g}$ *with respect to the Killing form.*

PROOF. To see that $\mathfrak{r} = \mathrm{Rad}(\mathfrak{g})$ is contained in $\mathscr{D}\mathfrak{g}^\perp$, i.e., that $\mathscr{D}\mathfrak{g}$ is perpendicular to $\mathfrak{r}$, let $X, Y \in \mathfrak{g}$ and $Z \in \mathfrak{r}$. Recalling that $B([X, Y], Z) = B(X, [Y, Z])$, it suffices to show that $B(X, [Y, Z]) = 0$. Let $\mathfrak{h}$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{r}$ and X. Then $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{r}$, so $\mathfrak{h}$ is solvable, so by Lie's theorem, under the adjoint action, $\mathfrak{h}$ acts on $\mathfrak{g}$ by upper-triangular matrices. By Lemma C.19, $[Y, Z]$ acts on $\mathfrak{g}$ by nilpotent transformations. It follows that $X \circ [Y, Z]$ also acts nilpotently on $\mathfrak{g}$, from which it follows that $B(X, [Y, Z]) = \operatorname{Tr}(X \circ [Y, Z]) = 0$, as required.

Since $\mathscr{D}\mathfrak{g}$ is a characteristic ideal, $(\mathscr{D}\mathfrak{g})^{\perp}$ is an ideal. It is solvable by Cartan's criterion (Proposition C.4), since

$$B(\mathscr{D}\mathfrak{g}^{\perp}, \mathscr{D}(\mathscr{D}\mathfrak{g}^{\perp})) \subset B(\mathscr{D}\mathfrak{g}^{\perp}, \mathscr{D}\mathfrak{g}) = 0.$$

It follows that $\mathscr{D}\mathfrak{g}^{\perp} \subset \mathfrak{r}$, which concludes the proof. □

Corollary C.23. *If* $\mathfrak{a}$ *is an ideal in a Lie algebra* $\mathfrak{g}$, *then*

$$\operatorname{Rad}(\mathfrak{a}) = \operatorname{Rad}(\mathfrak{g}) \cap \mathfrak{a}.$$

PROOF. Since $\operatorname{Rad}(\mathfrak{a})$ is a characteristic ideal of an ideal, it is an ideal of $\mathfrak{g}$. Since it is solvable, it must be contained in the radical of $\mathfrak{g}$. This shows the inclusion $\subset$; the opposition inclusion is clear since $\operatorname{Rad}(\mathfrak{g}) \cap \mathfrak{a}$ is a solvable ideal in $\mathfrak{a}$. □

Proposition C.24. *If D is a derivation of a Lie algebra* $\mathfrak{g}$, *then* $D(\operatorname{Rad}(\mathfrak{g}))$ *is contained in a nilpotent ideal of* $\mathfrak{g}$.

PROOF. Construct $\mathfrak{g}' = \mathfrak{g} \oplus \mathbb{C}$ as before, with $\xi = (0, 1)$. Since $\operatorname{Rad}(\mathfrak{g}) \subset \operatorname{Rad}(\mathfrak{g}')$, we have

$$D(\operatorname{Rad}(\mathfrak{g})) = [\xi, \operatorname{Rad}(\mathfrak{g})] \subset [\mathfrak{g}', \operatorname{Rad}(\mathfrak{g}')] \cap \mathfrak{g}.$$

By Lemma C.20, $[\mathfrak{g}', \operatorname{Rad}(\mathfrak{g}')]$ is a nilpotent ideal in $\mathfrak{g}'$, so its intersection with $\mathfrak{g}$ is also nilpotent. □

Just as with the notion of solvability, any Lie algebra $\mathfrak{g}$ contains a largest nilpotent ideal, usually called the *nil radical* of $\mathfrak{g}$, and denoted $\operatorname{Nil}(\mathfrak{g})$ or $\mathfrak{n}$. Proposition C.24 says that any derivation maps $\mathfrak{r}$ into $\mathfrak{n}$, which includes the result of Lemma C.20 that $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$. The existence of this ideal follows from:

Lemma C.25. *If* $\mathfrak{a}$ *and* $\mathfrak{b}$ *are nilpotent ideals in a Lie algebra* $\mathfrak{g}$, *then* $\mathfrak{a} + \mathfrak{b}$ *is also a nilpotent ideal.*

PROOF. An ideal $\mathfrak{a}$ is nilpotent iff there is a positive integer k so that all k-fold brackets $[X_1, [X_2, [\ldots, [X_{k-1}, X_k] \ldots]]]$ are zero when each X_i is in $\mathfrak{a}$. Equivalently, all m-fold brackets of m elements of $\mathfrak{g}$ are zero if at least k of them are in $\mathfrak{a}$. If k is chosen to work for $\mathfrak{a}$ and for $\mathfrak{b}$, it is easy to verify that $2k$ works for the sum $\mathfrak{a} + \mathfrak{b}$, since any bracket of $2k$ elements, each from $\mathfrak{a}$ or from $\mathfrak{b}$, contains at least k elements from $\mathfrak{a}$ or from $\mathfrak{b}$. □

Since $\operatorname{Nil}(\mathfrak{g}) \subset \operatorname{Rad}(\mathfrak{g})$, it follows from Proposition C.24 that $\operatorname{Nil}(\mathfrak{g})$ is a characteristic ideal of $\mathfrak{g}$. The same reasoning as in Corollary C.23 gives:

Corollary C.26. *If* $\mathfrak{a}$ *is an ideal in a Lie algebra* $\mathfrak{g}$, *then*

$$\mathrm{Nil}(\mathfrak{a}) = \mathrm{Nil}(\mathfrak{g}) \cap \mathfrak{a}.$$

If $\mathfrak{g}$ is a Lie algebra, its *universal enveloping algebra* $U = U(\mathfrak{g})$ is the quotient of the tensor algebra of $\mathfrak{g}$ modulo the two-sided ideal generated by all $X \otimes Y - Y \otimes X - [X, Y]$ for all X, Y in $\mathfrak{g}$. It is an associative algebra, with a map $\iota\colon \mathfrak{g} \to U$ such that

$$\iota([X, Y]) = [\iota(X), \iota(Y)] = \iota(X)\iota(Y) - \iota(Y)\iota(X),$$

and satisfying the universal property: for any linear map φ from $\mathfrak{g}$ to an associative algebra A such that $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$ for all X, Y, there is a unique homomorphism of algebras $\tilde{\varphi}\colon U \to A$ such that $\varphi = \tilde{\varphi} \circ \iota$. For example, a representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ determines an algebra homomorphism $\tilde{\rho}\colon U(\mathfrak{g}) \to \mathrm{End}(V)$. Conversely, any representation arises in this way.

We will need the following easy lemma:

Lemma C.27. *For any derivation* D *of a Lie algebra* $\mathfrak{g}$, *there is a unique derivation* $\tilde{D}$ *of the associative algebra* $U(\mathfrak{g})$ *such that* $\tilde{D} \circ \iota = \iota \circ D$.

Proof. Define an endomorphism of the tensor algebra of $\mathfrak{g}$ which is zero on the zeroth tensor power, and on the nth tensor power is

$$X_1 \otimes \cdots \otimes X_n \mapsto DX_1 \otimes X_2 \otimes \cdots \otimes X_n + X_1 \otimes DX_2 \otimes \cdots \otimes X_n + \cdots + X_1 \otimes X_2 \otimes \cdots \otimes DX_n.$$

This is well defined, since it is multilinear in each factor, and it is easily checked to be a derivation of the tensor algebra; denote it by D'. To see that D' passes to the quotient $U(\mathfrak{g})$ one checks routinely that it vanishes on generators for the ideal of relations. □

Exercise C.28. If D is an inner derivation by an element X in $\mathfrak{g}$, verify that $\tilde{D}$ is the inner derivation by the element $\iota(X)$.

It is a fact that the canonical map ι embeds $\mathfrak{g}$ in $U(\mathfrak{g})$. The *Poincaré–Birkhoff–Witt* theorem asserts that, in fact, if $U(\mathfrak{g})$ is filtered with the nth piece generated by all products of at most n products of elements of $\iota(\mathfrak{g})$, then the associated graded ring is the symmetric algebra on $\mathfrak{g}$. Equivalently, if $X_1, \ldots, X_r$ is a basis for $\mathfrak{g}$, then the monomials $X_1^{i_1} \cdot \ldots \cdot X_r^{i_r}$ form a basis for $U(\mathfrak{g})$. We do not need this theorem, but we will use the fact that these monomials generate $U(\mathfrak{g})$; this follows by a simple induction, using the equations $X_i \cdot X_j - X_j \cdot X_i = [X_i, X_j]$ to rearrange the order in products.

APPENDIX D

Cartan Subalgebras

Our task here is to prove the basic general facts that were stated in Lecture 14 about the decomposition of a semisimple Lie algebra $\mathfrak{g}$ into a Cartan algebra $\mathfrak{h}$ and a sum of root spaces $\mathfrak{g}_\alpha$, including the existence of such $\mathfrak{h}$ and its uniqueness up to conjugation.

§D.1. The Existence of Cartan Subalgebras

Note that if we have a decomposition as in Lecture 14, and H is any element of $\mathfrak{h}$ such that $\alpha(H) \neq 0$ for all roots α, then $\mathfrak{h}$ is determined by H: $\mathfrak{h} = \mathfrak{c}(H)$, where

$$\mathfrak{c}(H) = \{X \in \mathfrak{g}: [H, X] = 0\}. \qquad (D.1)$$

The elements of $\mathfrak{h}$ with this property are called *regular*. They form a Zariski open subset of $\mathfrak{h}$: the complement of the union of the hyperplanes defined by the equations $\alpha = 0$. In particular, regular elements are dense in $\mathfrak{h}$. If $H \in \mathfrak{h}$ is not regular, then $\mathfrak{c}(H)$ is larger than $\mathfrak{h}$, since it contains other root spaces. Note that all elements of $\mathfrak{h}$ are also semisimple, i.e., they are equal to their semisimple parts.

Of course, this discussion depends on knowing the decomposition which we are trying to prove. But it suggests one way to construct and characterize

Cartan subalgebras: they should be subalgebras of the form $\mathfrak{c}(H)$ for some semisimple element H, that are minimal in some sense. We can measure this minimality simply by dimension.

Definition D.2. The *rank* n of a semisimple Lie algebra $\mathfrak{g}$ is the minimum of the dimension of $\mathfrak{c}(H)$ as H varies over all semisimple elements of $\mathfrak{g}$. A semisimple element H is called *regular* if $\mathfrak{c}(H)$ has dimension n. A *Cartan subalgebra* of $\mathfrak{g}$ is an abelian subalgebra all of whose elements are semisimple, and that is not contained in any larger such subalgebra. Our first main goal is

Proposition D.3. *If H is regular, then $\mathfrak{c}(H)$ is a Cartan subalgebra.*

For any semisimple element H, $\mathfrak{g}$ decomposes into eigenspaces for the adjoint action of H:

$$\mathfrak{g} = \bigoplus_{\lambda} \mathfrak{g}_\lambda(H) = \mathfrak{c}(H) \oplus \bigoplus_{\lambda \neq 0} \mathfrak{g}_\lambda(H), \tag{D.4}$$

where $\mathfrak{g}_\lambda(H) = \{X \in \mathfrak{g}: [H, X] = \lambda X\}$, and $\mathfrak{c}(H) = \mathfrak{g}_0(H)$. There is a similar decomposition even if H (or $\mathfrak{g}$) is not semisimple, but replacing the eigenspace by $\mathfrak{g}_\lambda(H) = \{X \in \mathfrak{g}: (\mathrm{ad}(H) - \lambda I)^k(X) = 0 \text{ for large } k\}$.

Exercise D.5. Without assuming that H is semisimple, show that $[\mathfrak{g}_\lambda(H), \mathfrak{g}_\mu(H)] \subset \mathfrak{g}_{\lambda+\mu}(H)$, by proving the identity

$$\begin{aligned}&(\mathrm{ad}(H) - (\lambda + \mu)I)^k([X, Y]) \\ &\quad = \sum_{j=0}^{k} \binom{k}{j} [(\mathrm{ad}(H) - \lambda I)^j(X), (\mathrm{ad}(H) - \mu I)^{k-j}(Y)]\end{aligned}$$

Let us (temporarily) call an arbitrary element $H \in \mathfrak{g}$ regular if $\dim(\mathfrak{g}_0(H)) \leq \dim(\mathfrak{g}_0(X))$ for all $X \in \mathfrak{g}$.

Lemma D.6. *If H is regular, then $\mathfrak{g}_0(H)$ is abelian.*

PROOF. Consider how the Killing form B respects the decomposition (D.4)—again knowing what to expect from Lecture 14. If Y is in $\mathfrak{g}_\lambda(H)$ with $\lambda \neq 0$, then $\mathrm{ad}(Y)$ maps each eigenspace to a different eigenspace (by Exercise D.5), as does $\mathrm{ad}(Y) \circ \mathrm{ad}(X)$ for $X \in \mathfrak{g}_0(H)$. The trace of such an endomorphism is zero, i.e., $B(X, Y) = 0$ for such X and Y.

Because $\mathfrak{g}$ is semisimple, B is nondegenerate. Since we have shown that $\mathfrak{g}_0(H)$ is perpendicular to the other weight spaces, it follows that the restriction of B to $\mathfrak{g}_0(H)$ is nondegenerate.

Consider the Jordan decomposition $X = X_s + X_n$ of an element X in $\mathfrak{g}_0(H)$. Since $\mathrm{ad}(X_n) = \mathrm{ad}(X)_n$ is nilpotent, X_n belongs to $\mathfrak{g}_0(H)$, so $X_s = X - X_n$ does also. Then $\mathrm{ad}(X_s) = \mathrm{ad}(X)_s$ is nilpotent and semisimple on $\mathfrak{g}_0(H)$, so it vanishes there. But this already shows that $\mathrm{ad}(X) = \mathrm{ad}(X_s) + \mathrm{ad}(X_n)$ is a nilpotent endomorphism of $\mathfrak{g}_0(H)$ for any $X \in \mathfrak{g}_0(H)$. Hence, by Engel's theorem, $\mathfrak{g}_0(H)$ is nilpotent, so by Lie's theorem $\mathfrak{g}$ has a basis in which the endomorphisms $\mathrm{ad}(X)$ are upper-triangular for all $X \in \mathfrak{g}_0(H)$. It follows that for any elements in $\mathfrak{g}_0(H)$, the trace of products of their adjoint actions on $\mathfrak{g}$ is independent of the order of composition. In particular, for $X, Y, Z \in \mathfrak{g}_0(H)$, the trace of $\mathrm{ad}([X, Y]) \circ \mathrm{ad}(Z)$ on $\mathfrak{g}$ is zero, i.e., $B([X, Y], Z]) \equiv 0$. But since B is nondegenerate on $\mathfrak{g}_0(H)$, $[X, Y] = 0$, so $\mathfrak{g}_0(H)$ is abelian. □

It follows immediately that $\mathfrak{g}_0(H)$ is not contained in any larger abelian subalgebra, since any element that commutes with H is in $\mathfrak{g}_0(H)$ by definition. To finish the proof of the proposition we must prove the following lemma, which also shows that the temporary definition of regular agrees with the first one:

Lemma D.7. *If H is regular, then any element of $\mathfrak{g}_0(H)$ is semisimple.*

Proof. We saw that if X is in $\mathfrak{g}_0(H)$ then X_n is also. Using the same basis as in the preceding proof, we see that $\mathrm{ad}(X_n)$ has a strictly upper-triangular matrix. Hence, $B(X_n, Y) = \mathrm{Tr}(\mathrm{ad}(X_n) \circ \mathrm{ad}(Y)) = 0$ for all Y in $\mathfrak{g}_0(H)$. By the nondegeneracy again, $X_n = 0$, as required. □

It follows from Lemma D.6 that if H is regular, and X is in $\mathfrak{g}_0(H)$, then $\mathfrak{g}_0(X)$ contains $\mathfrak{g}_0(H)$, and they are equal exactly when X is also regular.

Problem D.8*. Prove that if H is regular in any Lie algebra, then $\mathfrak{g}_0(H)$ is a nilpotent Lie algebra.

Exercise D.9. Show that a subalgebra is a Cartan subalgebra if and only if it consists entirely of semisimple elements and is contained in no larger subalgebra with this property.

§D.2. On the Structure of Semisimple Lie Algebras

Let $\mathfrak{h}$ be a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Under the adjoint representation it consists of commuting semisimple endomorphisms. It is then a standard linear algebra fact that this action is simultaneously diagonalizable:

$$\mathfrak{g} = \bigoplus \mathfrak{g}_\alpha, \tag{D.10}$$

where the eigenspaces are parametrized by some set of linear forms $\alpha \in \mathfrak{h}^*$, including $\alpha = 0$, and where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}\colon [H, X] = \alpha(H) \cdot X \text{ for all } H \in \mathfrak{h}\}.$$

In particular, $\mathfrak{g}_0$ is the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$. The nonzero α are called *roots*.

Lemma D.11. $\mathfrak{h} = \mathfrak{g}_0$.

Proof. Since $\mathfrak{h}$ is abelian, $\mathfrak{h}$ is contained in $\mathfrak{g}_0$. If $\mathfrak{h}$ corresponds to a regular element H, i.e., $\mathfrak{h} = \mathfrak{g}_0(H)$, anything that commutes with H must be in $\mathfrak{h}$, so $\mathfrak{g}_0$ is contained in $\mathfrak{h}$. □

If $\mathfrak{h}$ is constructed from the regular element H, then by definition $\mathfrak{g}_\lambda(H)$ is the direct sum of those $\mathfrak{g}_\alpha$ for which $\alpha(H) = \lambda$. Note that the decomposition (D.10) may be finer than (D.4), but that if H is chosen to be an element of $\mathfrak{h}$ such that the $\alpha(H)$ are distinct for distinct roots α, then the decompositions coincide.

Our next task is to study the other eigenspaces $\mathfrak{g}_\alpha$. As before, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. It follows that if $\alpha + \beta \neq 0$, and if $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_\beta$, then $\mathrm{ad}(X) \circ \mathrm{ad}(Y)$ is nilpotent, so its trace is zero, i.e.,

$$\textit{If } \alpha + \beta \neq 0, \textit{ then } B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0. \tag{D.12}$$

Now for any root α, if $-\alpha$ were not a root, this implies $\mathfrak{g}_\alpha$ is perpendicular to all $\mathfrak{g}_\beta$ (including $\beta = 0$), which would contradict the nondegeneracy of B. So we get one of the facts asserted in Lecture 14:

$$\textit{If } \alpha \textit{ is a root, then } -\alpha \textit{ is also a root.} \tag{D.13}$$

Moreover, the pairing $B\colon \mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$ is nondegenerate. Another fact also follows easily:

$$\textit{The roots } \alpha \textit{ span } \mathfrak{h}^*. \tag{D.14}$$

For if not there would be a nonzero $X \in \mathfrak{h}$ with $\alpha(X) = 0$ for all roots α, which means that $[X, Y] = 0$ for all Y in all $\mathfrak{g}_\alpha$. But then X is in the center of $\mathfrak{g}$, which is zero by semisimplicity of $\mathfrak{g}$.

Now let α be a root, let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, and take any $H \in \mathfrak{h}$. Then

$$B(H, [X, Y]) = B([H, X], Y) = \alpha(H)B(X, Y). \tag{D.15}$$

This cannot be zero for all H, X, and Y without contradicting what we have just proved. In particular,

$$\textit{For any root } \alpha,\ [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \neq 0. \tag{D.16}$$

Let $T_\alpha \in \mathfrak{h}$ be the element dual to α via the pairing B on $\mathfrak{h}$, i.e., characterized by the identity $B(T_\alpha, H) = \alpha(H)$ for all H in $\mathfrak{h}$. We claim next that

$$[X, Y] = B(X, Y)T_\alpha \quad \text{for all } X \in \mathfrak{g}_\alpha,\ Y \in \mathfrak{g}_{-\alpha}. \tag{D.17}$$

To see it, pair both sides with an arbitrary element H of $\mathfrak{h}$. Using (D.15), we have

$$B(H, B(X, Y)T_\alpha) = B(H, T_\alpha)B(X, Y) = \alpha(H)B(X, Y) = B(H, [X, Y]),$$

as required. Next we show that

$$\alpha(T_\alpha) \neq 0. \tag{D.18}$$

Suppose this were false. Choose $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ such that $B(X, Y) = c \neq 0$. Then $[X, Y] = cT_\alpha$, so X, Y, and T_α span a Lie subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. If $\alpha(T_\alpha) = 0$, $\mathfrak{s}$ is solvable. Since $[X, Y] \in \mathscr{D}\mathfrak{s}$, it follows that $\text{ad}([X, Y])$ is a nilpotent endomorphism of $\mathfrak{g}$. But then T_α is nilpotent; but all elements of $\mathfrak{h}$ are semi-simple, so $T_\alpha = 0$, a contradiction. This gives another claim from Lecture 14:

$$\textit{For any root } \alpha,\ [[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}], \mathfrak{g}_\alpha] \neq 0. \tag{D.19}$$

For with X and Y as above, $[[X, Y], X] = c \cdot [T_\alpha, X] = c \cdot \alpha(T_\alpha)X \neq 0$.

The last remaining fact about root spaces left unproved from Lecture 14 is

$$\textit{For any root } \alpha,\ \mathfrak{g}_\alpha \textit{ is one-dimensional.} \tag{D.20}$$

By what we have seen, we can find $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$, so that $H = [X, Y] \neq 0$, and $\alpha(H) \neq 0$. Adjusting by scalars, they generate a subalgebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}_2\mathbb{C}$, with standard basis H, X, Y, so in particular $\alpha(H) = 2$. Consider the adjoint action of $\mathfrak{s}$ on the sum $V = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_{k\alpha}$, the sum over all nonzero complex multiples $k\alpha$ of α. From what we know about the weights of representations of $\mathfrak{s}$, the only k that can occur are integral multiples of $\frac{1}{2}$.

Now $\mathfrak{s}$ acts trivially on $\text{Ker}(\alpha) \subset \mathfrak{h} \subset V$, and it acts irreducibly on $\mathfrak{s} \subset V$. Together these cover the zero weight space $\mathfrak{h}$, since H is not in $\text{Ker}(\alpha)$. So the only even weights occurring can be 0 and ± 2. In particular,

$$2\alpha \textit{ cannot be a root.} \tag{D.21}$$

But this implies that $\frac{1}{2}\alpha$ cannot be a root, which says that 1 is not a weight occurring in V. But then there can be no other representations occurring in V, i.e., $V = \text{Ker}(\alpha) \oplus \mathfrak{s}$, which proves (D.20). □

§D.3. The Conjugacy of Cartan Subalgebras

We show that any two Cartan subalgebras are conjugate by an inner automorphism of the adjoint subgroup of $\text{Aut}(\mathfrak{g})$. Fix one Cartan subalgebra $\mathfrak{h}$, and consider the decomposition (D.10). For any element X in a root space $\mathfrak{g}_\alpha$, $\text{ad}(X) \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent, as we have seen, so its exponential $\exp(\text{ad}(X)) \in GL(\mathfrak{g})$ is just a finite polynomial in $\text{ad}(X)$. Set

$$e(X) = \exp(\text{ad}(X)).$$

Let $E(\mathfrak{h})$ be the subgroup of $\text{Aut}(\mathfrak{g})$ generated by all such $e(X)$. We want to prove now that this group is independent of the choice of $\mathfrak{h}$, and that all Cartan subalgebras are conjugate by elements in this group. (We will see in the next section that $E(\mathfrak{h})$ is the connected component of $\text{Aut}(\mathfrak{g})$, i.e., that it is the adjoint group.) The proof will be a kind of complex algebraic analogue of the corresponding argument for compact tori that was sketched in Lecture 26.

Theorem D.22. *Let* $\mathfrak{h}$ *and* $\mathfrak{h}'$ *be two Cartan subalgebras of* $\mathfrak{g}$. *Then* (i) $E(\mathfrak{h}) = E(\mathfrak{h}')$, *and* (ii) *there is an element* $g \in E = E(\mathfrak{h})$ *so that* $g(\mathfrak{h}) = \mathfrak{h}'$.

Proof. Fix a Cartan subalgebra $\mathfrak{h}$. Let $\alpha_1, \ldots, \alpha_r$ be its roots. Consider the mapping

$$F: \mathfrak{g}_{\alpha_1} \times \cdots \times \mathfrak{g}_{\alpha_r} \times \mathfrak{h} \to \mathfrak{g}$$

defined by $F(X_1, \ldots, X_r, H) = e(X_1) \circ \cdots \circ e(X_r)(H)$. Note that F is a polynomial mapping from one complex vector space to another of the same dimension. We want to show that not only is the image of F dense, but that, if $\mathfrak{h}_{\text{reg}}$ denotes the set of regular elements in $\mathfrak{h}$, then

$$F(\mathfrak{g}_{\alpha_1} \times \cdots \times \mathfrak{g}_{\alpha_r} \times \mathfrak{h}_{\text{reg}}) \textit{ contains a Zariski open set,} \tag{D.23}$$

i.e., it contains the complement of a hypersurface defined by a polynomial equation.

Suppose that this claim is proved. It follows that for any other Cartan subalgebra $\mathfrak{h}'$, the corresponding image also contains a Zariski open set. But two nonempty Zariski open sets always meet. In this case this means $E(\mathfrak{h}) \cdot \mathfrak{h}_{\text{reg}}$ meets $E(\mathfrak{h}') \cdot \mathfrak{h}'_{\text{reg}}$. That is, there are $g \in E(\mathfrak{h})$, $H \in \mathfrak{h}_{\text{reg}}$, $g' \in E(\mathfrak{h}')$, $H' \in \mathfrak{h}'_{\text{reg}}$ such that $g(H) = g'(H')$. But then since H and H' are regular,

$$g(\mathfrak{h}) = g(\mathfrak{g}_0(H)) = \mathfrak{g}_0(g(H)) = \mathfrak{g}_0(g'(H')) = g'(\mathfrak{g}_0(H')) = g'(\mathfrak{h}').$$

This proves the conjugacy of $\mathfrak{h}$ and $\mathfrak{h}'$. And since

$$E(\mathfrak{h}) = gE(\mathfrak{h})g^{-1} = E(g(\mathfrak{h})) = E(g'(\mathfrak{h}')) = g'E(\mathfrak{h}')(g')^{-1} = E(\mathfrak{h}'),$$

both statements of the theorem are proved. □

To prove (D.23), we use a special case of a very general fact from basic algebraic geometry: if $F: \mathbb{C}^N \to \mathbb{C}^N$ is a polynomial mapping whose derivative $dF_*|_P$ is invertible at some point P, then for any nonempty Zariski open set $U \subset \mathbb{C}^N$, $F(U)$ contains a nonempty Zariski open set. For the proof we refer to any basic algebraic geometry text, e.g., [Ha], or to [Bour, VI, App. A]. So it suffices to show that $dF_*|_P$ is surjective at a point $P = (0, \ldots, 0, H)$, where $H \in \mathfrak{h}_{\text{reg}}$. This is a simple calculation:

Exercise D.24*. Show that $dF_*|_P(0, \ldots, 0, Z) = Z$ for $Z \in \mathfrak{h}$, and that $dF_*|_P(0, \ldots, 0, Y, 0, \ldots, 0, 0) = \text{ad}(Y)(H) = -\text{ad}(H)(Y)$ for $Y \in \mathfrak{g}_{\alpha_i}$. Conclude that the image of $dF_*|_P$ contains $\mathfrak{h}$ and each root space, so $dF_*|_P$ is surjective. □

We remark that although this section, like the preceding appendix, was written for complex Lie algebras, a simple "base change" argument shows that the results extend to Lie algebras over any algebraically closed field of characteristic zero. Some, such as Cartan's criterion, then follow over any field of characteristic zero, by extending to an algebraic closure.

§D.4. On the Weyl Group

In this section we complete the proofs of some of the general facts about the Weyl group that were stated in Lectures 14 and 21. The notation will be as in those sections: $\mathbb{E}$ is the real space generated by the roots R; $\mathfrak{W}$ is the Weyl group, generated by the involutions W_α of $\mathbb{E}$ determined by

$$W_\alpha(\beta) = \beta - \beta(H_\alpha)\alpha = \beta - 2\frac{(\beta, \alpha)}{(\alpha, \alpha)}\alpha,$$

where (,) denotes the Killing form (or any inner product invariant for the Weyl group). We consider a decomposition

$$R = R^+ \cup R^-$$

into positive and negative roots, given by some $l\colon \mathbb{E} \to \mathbb{R}$ as in Lecture 14, and we let $S \subset R^+$ be the set of simple roots for this decomposition. Note that for any W in the Weyl group,

$$R = W(R^+) \cup W(R^-)$$

is the decomposition into positive and negative roots for the linear map $l \circ W^{-1}$. We want to show that every decomposition arises this way. To prove this we need some simple variations of the ideas in §21.1.

Lemma D.25. *If α is a simple root, then W_α permutes all the other positive roots, i.e., W_α maps $R^+\setminus\{\alpha\}$ to itself.*

Proof. This follows from the expression of positive roots as sums $\beta = \sum m_i\alpha_i$, with the α_i simple, and the m_i non-negative integers. If $\alpha = \alpha_i$, $W_\alpha(\beta)$ differs from β only by an integral multiple of α_i. If $\beta \neq \alpha_i$, $W_\alpha(\beta)$ still has some positive coefficients, so it must be a positive root. □

Let $\mathfrak{W}_0$ be the subgroup of $\mathfrak{W}$ generated by the W_α, as α varies over the simple roots. (We will soon see that $\mathfrak{W}_0 = \mathfrak{W}$.)

Lemma D.26. *Any root β can be written in the form $\beta = W(\alpha)$ for some $\alpha \in S$ and $W \in \mathfrak{W}_0$. In particular, $R = \mathfrak{W}(S)$.*

Proof. It suffices to do this for positive roots, since $\mathfrak{W}_0(\alpha) = \mathfrak{W}_0 W_\alpha(\alpha) = -\mathfrak{W}_0(\alpha)$ for any $\alpha \in S$. If β is positive but not simple, write $\beta = \sum m_i\alpha_i$ as above, and induct on the level $\sum m_i$. As in the previous lemma, there is a simple root γ so that $W_\gamma(\beta)$ is a positive root of lower level. By induction, $W_\gamma(\beta) = W(\alpha)$ for $\alpha \in S$ and $W \in \mathfrak{W}_0$, so $\beta = W_\gamma W(\alpha)$, as required. □

Lemma D.27. *The Weyl group is generated by the reflections in the simple roots, i.e., $\mathfrak{W} = \mathfrak{W}_0$.*

PROOF. Given a root β, we must show that W_β is in $\mathfrak{W}_0$. By the preceding lemma, write $\beta = U(\alpha)$ for some $U \in \mathfrak{W}_0$, $\alpha \in S$. Then

$$W_\beta = W_{U(\alpha)} = U \cdot W_\alpha \cdot U^{-1}, \tag{D.28}$$

since both sides act the same on β and $\beta^\perp$. □

Proposition D.29. *The Weyl group acts simply transitively on the set of decompositions of R into positive and negative roots.*

PROOF. For the transitivity, suppose $R = Q^+ \cup Q^-$ is another decomposition. We induct on the number of roots that are in R^+ but not in Q^+. If this number is zero, then $R^+ = Q^+$. Otherwise there must be some simple root α that is not in Q^+. It suffices to prove that $W_\alpha(Q^+)$ has more roots in common with R^+ than Q^+ does, for then by induction we can write $W_\alpha(Q^+) = W(R^+)$ for some $W \in \mathfrak{W}$, so $Q^+ = W_\alpha W(R^+)$, as required. In fact, we have by Lemma D.25,

$$W_\alpha(Q^+) \cap R^+ \supset W_\alpha(Q^+ \cap R^+) \cup \{\alpha\} = W_\alpha(Q^+ \cap R^+ \cup \{-\alpha\}),$$

and this proves the assertion.

For simple transitivity, we must show that if an element W in the Weyl group takes R^+ to itself, then it must be the identity. If not, write W as a product of reflections in simple roots,

$$W = W_1 \cdot \ldots \cdot W_r,$$

with r minimal, with W_i the reflection in the simple root β_i. Let $\alpha = \beta_r$. It suffices to show that

$$W_1 \cdot \ldots \cdot W_r = W_1 \cdot \ldots \cdot W_{s-1} W_{s+1} \cdot \ldots \cdot W_{r-1}$$

for some s, $1 \le s \le r-2$. Let $U_s = W_{s+1} \cdot \ldots \cdot W_{r-1}$. This equation is equivalent to the equation $W_s U_s W_r = U_s$, or $U_s W_r U_s^{-1} = W_s$, or $U_s(\alpha) = \beta_s$ (since by (D.28), $W_{U(\alpha)} = U W_\alpha U^{-1}$).

To finish the proof we must find an s so that $U_s(\alpha) = \beta_s$. Note that $U_{r-2}(\alpha) = W_{r-1}(\alpha)$ is a positive root (by Lemma D.25, since $\beta_{r-1} \neq \alpha$). On the other hand, the hypothesis implies that

$$U_0(\alpha) = W_1 \cdot \ldots \cdot W_{r-1}(\alpha) = W_1 \cdot \ldots \cdot W_r(-\alpha) = -W(\alpha)$$

is a negative root. So there must be some s with $1 \le s \le r-2$ such that $U_s(\alpha)$ is positive and $U_{s-1}(\alpha)$ is negative. This means that W_s takes the positive root $U_s(\alpha)$ to the negative root $U_{s-1}(\alpha)$. But by Lemma D.25 again, this can happen only if W_s is the reflection in the root $U_s(\alpha)$, i.e., $\beta_s = U_s(\alpha)$. □

The simple roots S for a decomposition $R = R^+ \cup R^-$ are called a *basis* for the roots. Since S and R^+ determine each other, the proposition is equivalent to the assertion that *the Weyl group acts simply transitively on the set of bases.*

Exercise D.30. For $W \in \mathfrak{W}$, set $l(W) = \#(R^+ \cap W(R^-))$. Show that W can be written as a product of $l(W)$ reflections in simple roots, but no fewer.

If Ω_α denotes the hyperplane in $\mathbb{E}$ perpendicular to the root α, the (closed) *Weyl chambers* are the closures of the connected components of the complement $\mathbb{E}\setminus\bigcup\Omega_\alpha$ of these hyperplanes. For a decomposition $R = R^+ \cup R^-$ with simple roots S, the set

$$\mathscr{W} = \{\beta \in \mathbb{E}: (\beta, \alpha) \geq 0, \forall\alpha \in R^+\} = \{\beta \in \mathbb{E}: (\beta, \alpha) \geq 0, \forall\alpha \in S\}$$

is one of these Weyl chambers. The fact that every Weyl chamber arises this way follows from

Lemma D.31. *For any β in $\mathbb{E}$ there is some $W \in \mathfrak{W}$ such that $(W(\beta), \alpha) \geq 0$ for all $\alpha \in S$.*

Proof. Let ρ be half the sum of the positive roots. It follows from Lemma D.25 that $W_\alpha(\rho) = \rho - \alpha$ for any simple root α. Take W in $\mathfrak{W}$ to maximize the inner product $(W(\beta), \rho)$. Then for all $\alpha \in S$,

$$(W_\alpha W(\beta), \rho) = (W(\beta), W_\alpha \rho) = (W(\beta), \rho - \alpha) = (W(\beta), \rho) - (W(\beta), \alpha)$$

cannot be larger than $(W(\beta), \rho)$, so $(W(\beta), \alpha) \leq 0$. □

Thus, the orbit of one Weyl chamber by the Weyl group covers $\mathbb{E}$, so all Weyl chambers are conjugate to each other by the action of the Weyl group. So all arise by partitioning R into positive and negative roots. This partitioning is uniquely determined by the Weyl chamber. In fact, the walls of a Weyl chamber are the hyperplanes Ω_α as α varies over the n corresponding simple roots, $n = \dim(\mathbb{E})$. From the proposition we have:

Corollary D.32. *The Weyl group acts simply transitively on Weyl chambers.*

Exercise D.33*. Let $\mathfrak{G}$ be the group of automorphisms of $\mathbb{E}$ that map R to itself.

(i) Show that $\mathfrak{W}$ is a normal subgroup of $\mathfrak{G}$.
(ii) Let $\mathfrak{R}$ be the automorphisms in $\mathfrak{G}$ which map a given set of simple roots S to itself. Show that $\mathfrak{G}$ is a semidirect product of $\mathfrak{W}$ and $\mathfrak{R}$.
(iii) Show that $\mathfrak{R}$ is isomorphic to the group of automorphisms of the Dynkin diagram.
(iv) Compute $\mathfrak{R}$ for each of the simple groups.

Our next goal is to show that the lattice $\mathbb{Z}\{H_\alpha: \alpha \in R\} \subset \mathfrak{h}$ has a basis of elements H_α where α varies over the simple roots. This is analogous to the statement we have proved that the root lattice Λ_R in $\mathfrak{h}^*$ is generated by simple roots. The first statement can be deduced from the second, using the Killing form to map $\mathfrak{h}$ to $\mathfrak{h}^*$, $H \mapsto (H, -)$, where $(\ ,\ )$ is the Killing form. We saw in Lecture 14 that this map takes H_α to $\alpha' = (2/(\alpha, \alpha))\alpha$. Given a root system R in a Euclidean space $\mathbb{E}$, to each root α one can define its *coroot* α' in $\mathbb{E}$ by the formula

$$\alpha' = \frac{2}{(\alpha, \alpha)}\alpha.$$

Let $R' = \{\alpha' : \alpha \in R\}$ be the set of coroots. For any $0 \neq \alpha \in \mathfrak{h}$, set $\alpha' = (2/(\alpha, \alpha))\alpha$, and for any $\alpha, \beta \in \mathfrak{h}^*$, set $n_{\beta\alpha} = 2(\beta, \alpha)/(\alpha, \alpha)$. Let $R = R^+ \cup R^-$ be a decomposition of R into positive and negative roots, and let S be the corresponding set of simple positive roots.

Lemma D.34. (i) *The set R' of coroots forms a root system in $\mathbb{E}$.*

(ii) *The set $S' = \{\alpha' : \alpha \in S\}$ is a set of simple roots for R'.*

(iii) *For $\alpha, \beta \in S$, $n_{\beta'\alpha'} = n_{\alpha\beta}$.*

Proof. It is a straightforward calculation that $n_{\beta'\alpha'} = n_{\alpha\beta}$. It follows by another short calculation that if W_α denotes the reflection in the hyperplane perpendicular to α, then $W_{\alpha'}(\beta') = (W_\alpha(\beta))'$. The four defining properties of a root system specified in §21.1 follow immediately from this. It is clear that if R^+ is the set of roots in R that are positive for a functional l on $\mathbb{E}$, then $(R^+)' = \{\alpha' : \alpha \in R^+\}$ is the corresponding set of positive roots for R'. Roots in R^+ are those that can be written as a nonnegative linear combinations of roots in S, and this property characterizes S. Since α' is a positive multiple of α for any α, it follows that roots in $(R^+)'$ are those that can be written as non-negative linear combinations of roots in S', which proves (ii). □

The root system R' is called the *dual* of R.

Exercise D.35. Find the dual of each type of simple root system.

Proposition D.36. (i) *The elements H_α for $\alpha \in S$ generate the lattice $\mathbb{Z}\{H_\alpha : \alpha \in R\}$.*

(ii) *If $\omega_\alpha \in \mathfrak{h}$ are defined by the property that $\omega_\alpha(H_\beta) = \delta_{\alpha,\beta}$, then the elements ω_α generate the weight lattice Λ_W.*

(iii) *The nonnegative integral linear combinations of the fundamental weights ω_α are precisely the weights in $\mathscr{W} \cap \Lambda_W$, where $\mathscr{W}$ is the closed Weyl chamber corresponding to R^+.*

Proof. The isomorphism $\mathfrak{h} \to \mathfrak{h}^*$ given by the Killing form takes H_α to the coroot α'. By the lemma and the fact that all positive roots are sums of simple roots, the set $\{\alpha' : \alpha \in S\}$ spans the same lattice as $\{\alpha' : \alpha \in R\}$. This proves (i), and it follows that the weights are precisely those elements in $\mathfrak{h}$ that take integral values on the set $\{H_\alpha : \alpha \in S\}$. The rest of the proposition follows, noting that

$$\begin{aligned} \mathscr{W} &= \{\beta \in \mathbb{E} : \beta(H_\alpha) \geq 0 \text{ for all } \alpha \in R^+\} \\ &= \{\beta \in \mathbb{E} : \beta(H_\alpha) \geq 0 \text{ for all } \alpha \in S\}. \end{aligned}$$ □

If we identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by means of the Killing form, we can regard $\mathfrak{W}$ as a group of automorphisms of $\mathfrak{h}$. By means of this, the reflection W_α corresponding to a root α becomes the automorphism of $\mathfrak{h}$ which takes an element H to $H - \alpha(H) \cdot H_\alpha$. We have a last debt (Fact 14.11) to pay about the Weyl group:

Proposition D.37. *Every element of the Weyl group is induced by an automorphism of* $\mathfrak{g}$ *which maps* $\mathfrak{h}$ *to itself.*

Proof. It suffices to produce the generating involutions W_α in this way. The claim is that if X_α and Y_α are generators of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ as usual, then $\vartheta_\alpha = e(X_\alpha)e(-Y_\alpha)e(X_\alpha)$ is such an automorphism, where, as in the preceding section, we write $e(X)$ for $\exp(\mathrm{ad}(X))$. We must show that $\vartheta_\alpha(H) = H - \alpha(H) \cdot H_\alpha$ for all H in $\mathfrak{h}$. It suffices to do this for H with $\alpha(H) = 0$, and for $H = H_\alpha$, since such together span $\mathfrak{h}$. If $\alpha(H) = 0$, then $[X_\alpha, H] = [Y_\alpha, H] = 0$, so $\vartheta_\alpha(H) = H$, which takes care of this case. For $H = H_\alpha$, it suffices to calculate on the subalgebra $\mathfrak{s}_\alpha = \mathbb{C}\{H_\alpha, X_\alpha, Y_\alpha\} \cong \mathfrak{sl}_2\mathbb{C}$, and this is a simple calculation:

Exercise D.38. (a) For $\mathfrak{sl}_2\mathbb{C}$ with its standard basis, show that $\vartheta = e(X)e(Y)e(X)$ maps H to $-H$, X to $-Y$, and Y to $-X$.

(b) Show that if G is a Lie group with Lie algebra $\mathfrak{g}$, then ϑ_α is induced by the element $\exp(\frac{1}{2}\pi(X_\alpha - Y_\alpha))$ of G.

We need a refinement of the preceding calculation. For a root α and a nonzero complex number t, define two automorphisms of $\mathfrak{g}$:

$$\vartheta_\alpha(t) = e(t \cdot X_\alpha) \circ e(-(t)^{-1} \cdot Y_\alpha) \circ e(t \cdot X_\alpha)$$

and

$$\Phi_\alpha(t) = \vartheta_\alpha(t) \circ \vartheta_\alpha(-1).$$

Lemma D.39. *The automorphism* $\Phi_\alpha(t)$ *is the identity on* $\mathfrak{h}$, *and for any root* β, *it is multiplication by* $t^{\beta(H_\alpha)}$ *on* $\mathfrak{g}_\beta$.

Proof. Look first in $\mathfrak{sl}_2$, with $X = X_\alpha$, $Y = Y_\alpha$. It is simplest to calculate in the covering $\mathrm{SL}_2\mathbb{C}$ of the adjoint group. Here $\vartheta_\alpha(t)$ lifts to

$$\begin{aligned}\exp(tX) \cdot \exp(-t^{-1}Y) \cdot \exp(tX) &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix},\end{aligned}$$

so $\Phi_\alpha(t)$ lifts to

$$\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

To see how $\Phi_\alpha(t)$ acts on $\mathfrak{g}_\beta$, for $\beta \neq \pm\alpha$, it suffices to consider the action of the $\mathrm{SL}_2\mathbb{C}$ corresponding to $\mathfrak{s}_\alpha = \mathbb{C}\{H_\alpha, X_\alpha, Y_\alpha\}$ on the α-string through β, i.e., on $\bigoplus \mathfrak{g}_{\beta+k\alpha}$. We know that this is an irreducible representation of $SL_2\mathbb{C}$, and the weight of $\mathfrak{g}_\alpha$ is $\beta(H_\alpha)$. It follows that $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ acts by multiplication by $t^{\beta(H_\alpha)}$. Similarly on $\mathfrak{h}$ it acts by multiplication by $t^0 = 1$. □

Putting the preceding results together, we can give a description of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$. Let $E = E(\mathfrak{h})$ be the subgroup generated by elements $\exp(\mathrm{ad}(Z))$, as Z varies over root spaces $\mathfrak{g}_\alpha$, $\alpha \neq 0$, as in §D.3. Let G be the adjoint form of $\mathfrak{g}$, so we have

$$E \subset G \subset \mathrm{Aut}^0(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g}),$$

where $\mathrm{Aut}^0(\mathfrak{g})$ is the connected component of the identity.

Proposition D.40. *We have* $E = G = \mathrm{Aut}^0(\mathfrak{g})$, *and* $\mathrm{Aut}(\mathfrak{g})/\mathrm{Aut}^0(\mathfrak{g})$ *is isomorphic to the automorphism group of the Dynkin diagram.*

PROOF. Fix the Cartan algebra $\mathfrak{h}$ and positive roots R^+. Let $\mathrm{Aut}(\mathfrak{g})'$ be the group of automorphisms of $\mathfrak{g}$ that map $\mathfrak{h}$ to itself, and similarly denote by primes the intersections of subgroups with $\mathrm{Aut}(\mathfrak{g})'$. We leave it to the reader to construct a finite subgroup K of $\mathrm{Aut}(\mathfrak{g})'$ which maps isomorphically onto the automorphism group of the Dynkin diagram, and which meets G only in the identity element (see Exercise 22.25 for a direct case-by-case approach, or use (21.25)). It then suffices to prove that $\mathrm{Aut}(\mathfrak{g})$ is a semidirect product of E and K, i.e., that $\mathrm{Aut}(\mathfrak{g}) = E \cdot K$.

To see this, start with any element σ in $\mathrm{Aut}(\mathfrak{g})$. By Theorem D.22, there is a $\tau_1 \in E$ with $\sigma(\mathfrak{h}) = \tau_1(\mathfrak{h})$. Then $\sigma_1 = \tau_1^{-1} \cdot \sigma$ is in $\mathrm{Aut}(\mathfrak{g})'$. By Proposition D.29 and the proof of Proposition D.37 there is a $\tau_2 \in E'$ so that $\sigma_2 = \tau_2^{-1} \cdot \sigma_1$ maps R^+ to R^+. This element may permute the simple roots, but there is some $k \in K$ so that $\sigma_3 = \sigma_2 \cdot k^{-1}$ is the identity on the set of simple roots. Now σ_3 is the identity on $\mathfrak{h}$ and it is multiplication by some nonzero scalar c_β on each $\mathfrak{g}_\beta$. By the nonsingularity of the Cartan matrix there is some nonzero complex number t and some $\lambda \in \Lambda_R$ so that $c_\beta = t^{\lambda(H_\beta)}$ for every simple root β. From Lemma D.39 it follows that there is a τ in E' so that τ and σ_3 agree on each $\mathfrak{g}_\beta$ for each simple root β, and both are the identity on $\mathfrak{h}$. But it then follows from the uniqueness theorem (Claim 21.25) that $\sigma_3 = \tau$. Hence

$$\sigma = \tau_1 \cdot \tau_2 \cdot \sigma_3 \cdot k \in E \cdot K,$$

as required. □

Exercise D.41. Show that any two Borel subalgebras of a semisimple Lie algebra are conjugate.

APPENDIX E

Ado's and Levi's Theorems

§E.1. Levi's Theorem

The object of this section is to prove Levi's theorem:

Theorem E.1. *Let* $\mathfrak{g}$ *be a Lie algebra with radical* $\mathfrak{r}$. *Then there is a subalgebra* $\mathfrak{l}$ *of* $\mathfrak{g}$ *such that* $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{l}$.

PROOF. There are several simple reductions. First, we may assume there is no nonzero ideal of $\mathfrak{g}$ that is properly contained in $\mathfrak{r}$. For if $\mathfrak{a}$ were such an ideal, by induction on the dimension of $\mathfrak{g}$, $\mathfrak{g}/\mathfrak{a}$ would have a subalgebra complementary to $\mathfrak{r}/\mathfrak{a}$, and this subalgebra has the form $\mathfrak{l}/\mathfrak{a}$, with $\mathfrak{l}$ as required. In particular, we may assume $\mathfrak{r}$ is abelian, since otherwise $\mathscr{D}\mathfrak{r}$ is a proper ideal in $\mathfrak{r}$ which is an ideal in $\mathfrak{g}$ by Corollary C.23. We may also assume that $[\mathfrak{g}, \mathfrak{r}] = \mathfrak{r}$, for if $[\mathfrak{g}, \mathfrak{r}] = 0$ then the adjoint representation factors through $\mathfrak{g}/\mathfrak{r}$, and since $\mathfrak{g}/\mathfrak{r}$ is semisimple, the submodule $\mathfrak{r} \subset \mathfrak{g}$ has a complement, which is the required $\mathfrak{l}$.

Now $V = \mathfrak{gl}(\mathfrak{g})$ is a $\mathfrak{g}$-module via the adjoint representation: for $X \in \mathfrak{g}$ and $\varphi \in V$,

$$X \cdot \varphi = [\mathrm{ad}(X), \varphi] = \mathrm{ad}(X) \circ \varphi - \varphi \circ \mathrm{ad}(X).$$

In other words, for $X, Y \in \mathfrak{g}$ and $\varphi \in V$,

$$(X \cdot \varphi)(Y) = [X, \varphi(Y)] - \varphi([X, Y]). \tag{E.2}$$

The trick is to consider the following subspaces of V:

$$\begin{aligned}
C &= \{\varphi \in V: \varphi(\mathfrak{g}) \subset \mathfrak{r} \text{ and } \varphi|_{\mathfrak{r}} \text{ is multiplication by a scalar}\}\\
&\cup\\
B &= \{\varphi \in V: \varphi(\mathfrak{g}) \subset \mathfrak{r} \text{ and } \varphi(\mathfrak{r}) = 0\}\\
&\cup\\
A &= \{\mathrm{ad}(X): X \in \mathfrak{r}\}.
\end{aligned}$$

These are easily checked to be $\mathfrak{g}$-submodules of V, included in each other as indicated. And C/B is a trivial $\mathfrak{g}$-module of rank 1, i.e. $C/B = \mathbb{C}$, by taking φ in C to the scalar λ such that $\varphi|_{\mathfrak{r}} = \lambda \cdot I$. (Note that $C/B \neq 0$ since one can find an endomorphism of the vector space $\mathfrak{g}$ which is the identity on $\mathfrak{r}$ and zero on a vector space complement to $\mathfrak{r}$.) We claim also that

$$\mathfrak{g} \cdot C \subset B \quad \text{and} \quad \mathfrak{r} \cdot C \subset A. \tag{E.3}$$

To prove these let $\varphi \in C$, and assume the restriction of φ to $\mathfrak{r}$ is multiplication by the scalar c. If $X \in \mathfrak{g}$ and $Y \in \mathfrak{r}$, then by (E.2),

$$(X \cdot \varphi)(Y) = [X, cY] - c[X, Y] = 0,$$

so $X \cdot \varphi \in B$; this proves the first inclusion. If $X \in \mathfrak{r}$, and $Y \in \mathfrak{g}$, then $[X, \varphi(Y)] \in [\mathfrak{r}, \mathfrak{r}] = 0$, so

$$(X \cdot \varphi)(Y) = -\varphi([X, Y]) = [-cX, Y],$$

and $X \cdot \varphi = \mathrm{ad}(-cX)$ is in A, which proves the second inclusion.

This means that the map $C/A \to C/B = \mathbb{C}$ is a surjection of $\mathfrak{g}/\mathfrak{r}$-modules, which must split since $\mathfrak{g}/\mathfrak{r}$ is semisimple. In other words, there is an element φ in C such that $\varphi|_{\mathfrak{r}} = \mathrm{id}_{\mathfrak{r}}$ and $\mathfrak{g} \cdot \varphi$ is contained in A. Now let

$$\mathfrak{l} = \{X \in \mathfrak{g}: X \cdot \varphi = 0\}.$$

It is easy to check that $\mathfrak{l}$ is a subalgebra of $\mathfrak{g}$. We must verify: (i) $\mathfrak{l} \cap \mathfrak{r} = 0$; and (ii) $\mathfrak{g} = \mathfrak{l} + \mathfrak{r}$. For the first, if X is a nonzero element of the intersection, then, as we saw above, $X \cdot \varphi = \mathrm{ad}(-X)$, so $\mathrm{ad}(X) = 0$. Hence $[\mathfrak{g}, X] = 0$, so $\mathbb{C} \cdot X$ is a nonzero ideal in $\mathfrak{r}$, contradicting our assumptions. For (ii), let $X \in \mathfrak{g}$. Then $X \cdot \varphi$ is in A, so $X \cdot \varphi = \mathrm{ad}(Y)$ for some Y in $\mathfrak{r}$. We saw that $\mathrm{ad}(Y) = -Y \cdot \varphi$, so $(X + Y) \cdot \varphi = 0$, i.e., $X + Y$ belongs to $\mathfrak{l}$. Hence $X = (X + Y) - Y$ is in the sum of $\mathfrak{l}$ and $\mathfrak{r}$. □

This proves the existence of Levi subalgebras $\mathfrak{l}$ of any Lie algebra. We have no need to prove the companion fact that any two Levi subalgebras are conjugate, cf. [Bour, I, §6.8].

§E.2. Ado's Theorem

The goal is Ado's theorem that every Lie algebra is linear, i.e., is a subalgebra of $\mathfrak{gl}(V)$ for some vector space V, which is the same as saying it has a finite-dimensional faithful representation. As in the previous section, there are some easy steps, and then a clever argument is needed to create an appropriate representation.

We start, of course, with the adjoint representation, which is about the only representation we have for an abstract Lie algebra $\mathfrak{g}$. Since the kernel of the adjoint representation is the center $\mathfrak{c}$ of $\mathfrak{g}$, it suffices to find a representation of $\mathfrak{g}$ which is faithful on $\mathfrak{c}$. For then the sum of this representation and the adjoint representation is a faithful representation of $\mathfrak{g}$.

The abelian Lie algebra $\mathfrak{c}$ has a faithful representation by nilpotent matrices. For example, when $\mathfrak{c} = \mathbb{C}$ is one dimensional, one can take the representation $\lambda \mapsto \left(\begin{smallmatrix} 0 & \lambda \\ 0 & 0 \end{smallmatrix}\right)$; in general a direct sum of such representations will suffice.

We can choose a sequence of subalgebras

$$\mathfrak{c} = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_p = \mathfrak{n} \subset \mathfrak{g}_{p+1} \subset \cdots \subset \mathfrak{g}_q = \mathfrak{r} \subset \mathfrak{g}_{q+1} = \mathfrak{g},$$

each an ideal in the next, with $\mathfrak{n} = \mathrm{Nil}(\mathfrak{g})$ the largest nilpotent ideal of $\mathfrak{g}$, and $\mathfrak{r} = \mathrm{Rad}(\mathfrak{g})$ the largest solvable ideal; as in §9.1 we may assume $\dim(\mathfrak{g}_i/\mathfrak{g}_{i-1}) = 1$ for $i \le q$. The plan is to start with a faithful representation of $\mathfrak{g}_0$, and construct successively representations of each $\mathfrak{g}_i$ which are faithful on $\mathfrak{c}$. The conditions we will need to make this step are that $\mathfrak{g}_i = \mathfrak{g}_{i-1} \oplus \mathfrak{h}_i$ with $\mathfrak{g}_{i-1}$ a solvable ideal in $\mathfrak{g}_i$ and $\mathfrak{h}_i$ a subalgebra of $\mathfrak{g}_i$. We can achieve this by taking $\mathfrak{h}_i$ to be any one-dimensional vector space complementary to $\mathfrak{g}_{i-1}$ for $i \le q$. Similarly to go from $\mathfrak{r}$ to $\mathfrak{g}$, use Levi's theorem to write $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{h}$ for a subalgebra $\mathfrak{h}$.

Call a representation ρ of a Lie algebra $\mathfrak{g}$ a *nilrepresentation* if $\rho(X)$ is a nilpotent endomorphism for every X in $\mathrm{Nil}(\mathfrak{g})$. A stronger version of Ado's theorem is:

Theorem E.4. *Every Lie algebra has a faithful finite-dimensional nilrepresentation.*

The crucial step is:

Proposition E.5. *Let $\mathfrak{g}$ be a Lie algebra which is a direct sum of a solvable ideal $\mathfrak{a}$ and a subalgebra $\mathfrak{h}$. Let σ be a nilrepresentation of $\mathfrak{a}$. Then there is a representation ρ of $\mathfrak{g}$ such that*

$$\mathfrak{h} \cap \mathrm{Ker}(\rho) \subset \mathrm{Ker}(\sigma).$$

If $\mathrm{Nil}(\mathfrak{g}) = \mathrm{Nil}(\mathfrak{a})$ or $\mathrm{Nil}(\mathfrak{g}) = \mathfrak{g}$, then ρ may be taken to be a nilrepresentation.

Ado's theorem follows readily from this proposition. Starting with a faithful representation ρ_0 of $\mathfrak{c} = \mathfrak{g}_0$ by nilpotent matrices, one uses the proposition to construct successively nilrepresentations ρ_i of $\mathfrak{g}_i$. The displayed condition assures that they are all faithful on $\mathfrak{c}$. Note that if $i \le p$, $\mathrm{Nil}(\mathfrak{g}_i) = \mathfrak{g}_i$, while if $i > p$ we have $\mathrm{Nil}(\mathfrak{g}_i) = \mathrm{Nil}(\mathfrak{g}_{i-1}) = \mathfrak{n}$ by Corollary C.26, so the hypotheses assure that all representations can be taken to be nilrepresentations. □

Suppose $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ is a Lie algebra which is a direct sum of an ideal $\mathfrak{a}$ and a subalgebra $\mathfrak{h}$. Let $U = U(\mathfrak{a})$ be the universal enveloping algebra of $\mathfrak{a}$. Any

Y in $\mathfrak{a}$ determines a linear endomorphism L_Y of U, which is simply left multiplication by the image of Y in U. Any X in $\mathfrak{g}$ determines an inner derivation $Y \mapsto [X, Y]$ of $\mathfrak{a}$; let D_X be the corresponding derivation of U, cf. Lemma C.27. For each X in $\mathfrak{g}$ we define a linear mapping $T_X: U \to U$ by writing $X = Y + Z$ with Y in $\mathfrak{a}$ and Z in $\mathfrak{h}$, and setting

$$T_X = L_Y + D_Z.$$

A straightforward calculation shows that

$$T_{[X_1, X_2]} = T_{X_1} \circ T_{X_2} - T_{X_2} \circ T_{X_1}. \tag{E.6}$$

If $\mathfrak{gl}(U)$ denotes the infinite-dimensional Lie algebra of endomorphisms of U, with the usual bracket $[A, B] = A \circ B - B \circ A$, this means that the mapping $\mathfrak{a} \to \mathfrak{gl}(U)$, $X \mapsto T_X$, is a homomorphism of Lie algebras.

Suppose $\sigma: \mathfrak{a} \to \mathfrak{gl}(V)$ is a finite-dimensional representation of $\mathfrak{a}$. Let $\tilde{\sigma}: U \to \mathrm{End}(V)$ be the corresponding homomorphism of algebras, as in §C.3, and let I be the kernel of $\tilde{\sigma}$. The basic step is:

Lemma E.7. *Assume that $\mathfrak{a}$ is solvable. Suppose I is an ideal of $U = U(\mathfrak{a})$ satisfying the following two properties*: (i) *U/I is finite dimensional*; (ii) *the image of every element in* $\mathrm{Nil}(\mathfrak{a})$ *in U/I is nilpotent. Then there is an ideal $J \subset I$ of U satisfying properties* (i) *and* (ii), *and also* (iii) *for every derivation D of* $\mathfrak{a}$, *the corresponding derivation of U maps J into itself.*

Granting this lemma, we prove Proposition E.5 as follows. From the representation σ we constructed an ideal I in $U = U(\mathfrak{a})$, with $U/I \subset \mathrm{End}(V)$, so condition (i) is satisfied; the fact that σ is a nilrepresentation implies that condition (ii) also holds. Let J be an ideal whose existence is asserted in the lemma. Because of (iii), each of the endomorphisms T_X of U maps J into itself, and so determines an endomorphism $\overline{T_X}$ of U/J. By (E.6), the mapping $X \mapsto \overline{T_X}$ is a homomorphism of Lie algebras from $\mathfrak{g}$ to $\mathfrak{gl}(U/J)$. This is the representation ρ required in the proposition.

We first verify that $\mathrm{Ker}(\rho) \cap \mathfrak{a} \subset \mathrm{Ker}(\sigma)$. Note that if X is in $\mathfrak{a}$, then $\overline{T_X}$ is just left multiplication by X on U/J, so if $\rho(X)$ vanishes, the image of X in U must be in J; since $J \subset I$, X maps to zero in $U/I \subset \mathrm{End}(V)$, so $\sigma(X) = 0$, as required.

It remains to show that, under either of the additional hypotheses, ρ is a nilrepresentation. Note first that each X in $\mathfrak{a}$ acts on U/J by left multiplication, and if X is in $\mathrm{Nil}(\mathfrak{a})$, by (ii) its image in U/J is nilpotent. Thus $\rho(X)$ is nilpotent for every X in $\mathrm{Nil}(\mathfrak{a})$. In particular, this shows that ρ is a nilrepresentation when $\mathrm{Nil}(\mathfrak{g}) = \mathrm{Nil}(\mathfrak{a})$.

In the other case, $\mathfrak{g}$ is nilpotent, so $\mathfrak{a}$ is also nilpotent, and the preceding shows that $\rho(Y)$ is nilpotent for every Y in $\mathfrak{a}$. We need a slightly stronger assertion than this. Let $A \subset \mathrm{End}(U/J)$ be the associative algebra (with unit) generated by $\rho(\mathfrak{g})$, and let $P \subset A$ be the two-sided ideal generated by $\rho(\mathfrak{a})$. The claim is that P is a nilpotent ideal, i.e., that $P^k = P \cdot \ldots \cdot P = 0$ for some k. To see this, note that there is a k such that every product of k elements of $\rho(\mathfrak{a})$ is zero; this follows from Engel's theorem, putting the action in strictly upper-triangular form. To show that $P^k = 0$, we must show that any product of elements in $\rho(\mathfrak{g})$ which contains at least k members from $\rho(\mathfrak{a})$ is zero. But if x is in $\rho(\mathfrak{g})$ and y is in $\rho(\mathfrak{a})$, we have

$$x \cdot y = y \cdot x + [x, y],$$

and $[x, y]$ is in $\rho(\mathfrak{a})$, so terms from $\rho(\mathfrak{a})$ can be successively moved to the left until the product is a sum of products each beginning with k terms from $\rho(\mathfrak{a})$.

Now if $\mathfrak{g}$ is nilpotent, for any Z in $\mathfrak{h}$ (or in $\mathfrak{g}$), $\mathrm{ad}(Z)$ is a nilpotent endomorphism of $\mathfrak{g}$, and hence of $\mathfrak{a}$. By the Leibnitz rule for derivations, it follows that the corresponding derivation D_Z of U is nilpotent on any element, although the power required to annihilate an element may be unbounded. However, since U/J is finite dimensional, it follows readily that the induced derivation of U/J is nilpotent. In other words, $\rho(Z)$ is nilpotent for every Z in $\mathfrak{h}$. Given X in $\mathfrak{g}$, write $X = Y + Z$ with $Y \in \mathfrak{a}$ and $Z \in \mathfrak{h}$. Choose k as in the preceding paragraph, and choose l so that $\rho(Z)^l = 0$. It follows that $\rho(X)^{kl} = (\rho(Y) + \rho(Z))^{kl}$ vanishes, since, when the latter is expanded, each summand either has $\rho(Y)$ occurring at least k times, or else $\rho(Z)^l$ occurs somewhere in the product. □

To finish, we must prove Lemma E.7. Let Q be the two-sided ideal in the algebra U/I generated by the image of $\mathrm{Nil}(\mathfrak{a})$. Since U/I is generated by the image of $\mathfrak{a}$, the same argument as in the paragraph before last shows that $Q^k = 0$ for some k. Write $Q = K/I$ for an ideal K of U, and set $J = K^k$. Clearly $J \subset I$, and we claim that J satisfies the conditions (i)–(iii) of the lemma.

To see that J has finite codimension, let $x_1, \ldots, x_n$ be a basis for the image of $\mathfrak{a}$ in U, and choose monic polynomials p_i such that $p_i(x_i)$ is in K; this is possible since U/K is finite dimensional. Therefore, $p_i(x_i)^k$ is in J, so the images of the x_i satisfy monic equations in U/J. Since U is generated by the monomials $x_1^{i_1} \cdot \ldots \cdot x_r^{i_r}$, it follows readily that U/J is spanned by a finite number of these elements.

Property (ii) is clear from the construction, for if $x \in U$ is the image of an element of $\mathrm{Nil}(\mathfrak{a})$, some power x^p is in I by assumption, so x^{pk} is in $I^k \subset K^k = J$.

For (iii), if D is a derivation of $\mathfrak{a}$, since $\mathfrak{a}$ is solvable, it follows from Proposition C.24 that D maps $\mathfrak{a}$ into $\mathrm{Nil}(\mathfrak{a})$. The corresponding derivation of U therefore maps U into K, from which it follows that it maps $J = K^k$ to itself. □

As before, the results of this section also apply to real Lie algebras: if $\mathfrak{g}$ is real, a faithful representation (complex) representation of $\mathfrak{g} \otimes \mathbb{C}$ is automatically a faithful real representation, and embeds $\mathfrak{g}$ is some $\mathfrak{gl}_n\mathbb{R}$.

APPENDIX F

Invariant Theory for the Classical Groups

The object is to derive just enough invariant theory for the classical groups to verify the claims made in the text. We follow a classical, constructive approach, using an identity of Capelli.

§F.1. The Polynomial Invariants

Let $V = \mathbb{C}^n$, regarded as the standard representation of $\mathrm{GL}_n\mathbb{C}$, so of any of the subgroups $G = \mathrm{SL}_n\mathbb{C}$, $\mathrm{O}_n\mathbb{C}$, $\mathrm{SO}_n\mathbb{C}$, or $\mathrm{Sp}_n\mathbb{C}$ (for n even); $e_1, \ldots, e_n$ denotes a standard basis for V, compatible with one of the standard realizations of G. The goal is to find those polynomials $F(x^{(1)}, \ldots, x^{(m)})$ of m variables on V which are invariant by G. For example, if $Q\colon V \otimes V \to \mathbb{C}$ is the bilinear form determining the orthogonal or symplectic group, the polynomials $Q(x^{(i)}, x^{(j)})$ are invariants. In addition, if G is a subgroup of $\mathrm{SL}(V)$, the *bracket* $[x^{(1)}\, x^{(2)} \ldots x^{(n)}]$, given by the determinant,

$$[x^{(1)} \quad x^{(2)} \quad \ldots \quad x^{(n)}] = \det(x_j^{(i)}), \tag{F.1}$$

is an invariant of G. The *first fundamental theorem* of invariant theory for these groups asserts that any invariant is a polynomial function of these basic invariants. This is the goal of this appendix.

We denote by S^d the homogeneous polynomial functions of degree d on V, i.e., $S^d = \mathrm{Sym}^d(V^*)$. For an m-tuple $\mathbf{d} = (d_1, \ldots, d_m)$ of non-negative integers, let $S^{\mathbf{d}} = S^{d_1} \otimes \cdots \otimes S^{d_m}$ be the polynomials on $V^{\oplus m}$ which are homogeneous of degree d_i in the ith variable. Note that

$$\mathrm{Sym}^k(V^{\oplus m})^* = \bigoplus S^{\mathbf{d}},$$

the sum over all $\mathbf{d}$ with $d_1 + d_2 + \cdots + d_m = k$, which identifies elements of $S^{\mathbf{d}}$ with functions of m-tuples in V. We write $F(x^{(1)}, \ldots, x^{(m)})$ for such a polynomial, with usual abbreviations to $F(x)$ for $m = 1$, $F(x, y)$ for $m = 2$, $F(x, y, z)$ for $m = 3$.

When $m = 1$ we have already found the invariants: for $\mathrm{SL}_n\mathbb{C}$ and $\mathrm{Sp}_n\mathbb{C}$ all symmetric powers S^d are irreducible, so there are no invariants unless $d = 0$; for $\mathrm{SO}_n\mathbb{C}$ the kernel of the map $S^d \to S^{d-2}$ (contracting with the given quadratic form Q) is irreducible, so by induction one sees that there are no invariants if d is odd, whereas if d is even, the invariants are scalar multiples of the polynomial $Q(x, x)^{d/2}$. (These results will be proved again below.)

In theory one could follow procedures outlined in the text to decompose the tensor products of the known representations S^{d_i} to find out how the trivial representation occurs in $S^{\mathbf{d}}$. Except in small degrees and dimensions, however, this is rather impractical.

To describe the G-invariant polynomials in $S^{\mathbf{d}}$, we will carry out an induction, first with respect to the total degree $\sum d_i$, then with respect to the individual multidegrees ordered *antilexicographically*: $\mathbf{d}' < \mathbf{d}$ means that either $\sum d_i' < \sum d_i$ or $\sum d_i' = \sum d_i$ and the largest i for which d_i' and d_i differ has $d_i' < d_i$.

For integers i and j between 1 and m there is a canonical "polarization" map D_{ij} which takes a polynomial F of m variables to the polynomial

$$D_{ij}(F) = \sum_{k=1}^{n} x_k^{(i)} \frac{\partial F}{\partial x_k^{(j)}}. \tag{F.2}$$

This operator lowers the jth degree by 1, while it increases the ith degree by 1, i.e., it maps $S^{\mathbf{d}}$ to $S^{\mathbf{d}'}$, where $\mathbf{d}'$ is the same sequence of multi-indices as $\mathbf{d}$, but with $d_j' = d_j - 1$ and $d_i' = d_i + 1$; if $d_j = 0$ set $S^{\mathbf{d}'} = 0$. When $j = i$, note that by *Euler's formula*, D_{ii} is multiplication by d_i. Note also that these D_{ij} are derivations:

$$D_{ij}(F_1 \cdot F_2) = D_{ij}(F_1) \cdot F_2 + F_1 \cdot D_{ij}(F_2). \tag{F.3}$$

These maps may be described intrinsically in terms of the multilinear algebra of Appendix B, as follows. Since only two factors are involved, it suffices to look at the map D_{12} when there are only two factors. In this case the map

$$D_{12}\colon S^d \otimes S^e \to S^{d+1} \otimes S^{e-1}$$

is defined by

$$u_1 \cdot \ldots \cdot u_d \otimes w_1 \cdot \ldots \cdot w_e \mapsto \sum_{i=1}^{e} u_1 \cdot \ldots \cdot u_d \cdot w_i \otimes w_1 \cdot \ldots \cdot \hat{w}_i \cdot \ldots \cdot w_e.$$

Equivalently, D_{12} is the composite

$$S^d \otimes S^e \to S^d \otimes (S^1 \otimes S^{e-1}) = (S^d \otimes S^1) \otimes S^{e-1} \to S^{d+1} \otimes S^{e-1},$$

where the second is determined by the product $S^d \otimes S^1 \to S^{d+1}$ of symmetric powers, and the first by the dual map $S^e \to S^1 \otimes S^{e-1}$ (which takes $F(x)$ to $\sum_k x_k \otimes \partial F/\partial x_k$). This shows, if there were any doubt, that the D_{ij} are maps of $\mathrm{GL}(V)$-modules, i.e., that they are independent of choice of coordinates.

Note that $D_{ji} \circ D_{ij}$ maps $S^{\mathbf{d}}$ to itself. Explicitly, for $\mathbf{d} = (d, e)$,

$$\begin{aligned} D_{21} \circ D_{12}(F) &= \sum_k y_k \frac{\partial}{\partial x_k}\left(\sum_l x_l \frac{\partial F}{\partial y_l}\right) \\ &= \sum_k y_k \frac{\partial F}{\partial y_k} + \sum_{k,l} y_k x_l \frac{\partial^2 F}{\partial y_l \partial x_k} \\ &= e \cdot F + \sum_{k,l} y_k x_l \frac{\partial^2 F}{\partial y_l \partial x_k}. \end{aligned}$$

A first idea is that, if F is an invariant by a group $G \subset \mathrm{GL}(V)$, then $D_{ij}(F)$ will also be an invariant, and these invariants will be known by induction if $i < j$, so one can describe the possible $D_{ji} \circ D_{ij}(F)$ that arise. If one also knew the second term in the above expression for this, one could determine $e \cdot F$, which suffices to determine F, provided e is not zero.

In general, it is not evident how to proceed, but in case $\dim V = 2$, and $\mathbf{d} = (d, e)$, this can idea can be carried through as follows. Some of the terms in the second term also occur in the expression

$$[xy] \cdot \Omega(F) = (x_1 y_2 - x_2 y_1) \cdot \left(\frac{\partial^2 F}{\partial x_1 \partial y_2} - \frac{\partial^2 F}{\partial x_2 \partial y_1}\right).$$

The rest occur in

$$de \cdot F = d \cdot \left(\sum y_l \frac{\partial F}{\partial y_l}\right) = \sum x_k y_l \frac{\partial^2 F}{\partial x_k \partial y_l}.$$

Comparing the preceding three formulas gives the identity

$$(d + 1)e \cdot F = D_{21} \circ D_{12}(F) + [xy] \cdot \Omega(F). \tag{F.4}$$

From this identity it is easy to find all invariants for one of our subgroups of $\mathrm{GL}_2\mathbb{C}$ and for functions of two variables. We will do it for $G = \mathrm{SO}_2\mathbb{C}$, as it illustrates the ideas of the general case—even though G is not semisimple, and the results can be seen directly by identifying G with $\mathbb{C}^*$. We assume the simple case of functions of one variable has been checked: only multiples of $Q(x, x)^{d/2}$ are invariant. Suppose $F \in S^d \otimes S^e$ is an invariant of $G = \mathrm{SO}_2\mathbb{C}$, with $e > 0$. We claim that F is a polynomial in the bracket function $[xy]$ and the polynomials $Q(x, y)$, $Q(x, x)$, and $Q(y, y)$. Either directly or from the above identity one sees that $\Omega(F)$ is also an $\mathrm{SO}_2\mathbb{C}$-invariant, and by induction it is a polynomial in these basic polynomials. Similarly by the antilexicographic induction we know that $D_{12}(F)$ is a polynomial in the the basic invariants. It therefore suffices to verify that D_{21} preserves polynomials in the four basic invariants. By the derivation property (F.3) it is enough to compute the effect of D_{21} on the basic invariants, and this is easy:

$$D_{21}[xy] = 0, \qquad D_{21}Q(x, y) = Q(y, y),$$
$$D_{21}Q(x, x) = 2Q(x, y), \qquad D_{21}Q(y, y) = 0.$$

By (F.4) we conclude that $(d + 1)e \cdot F$ is a polynomial in the basic invariants, which concludes the proof.

This plan of attack, in fact, extends to find all polynomial invariants of all the classical subgroups of $\mathrm{GL}(V)$. What is needed is an appropriate generalization of the identity (F.4). About a century ago Capelli found such an identity. The clue is to write (F.4) in the more suggestive form

$$\begin{vmatrix} D_{11} + 1 & D_{12} \\ D_{21} & D_{22} \end{vmatrix} (F) = [xy] \cdot \Omega(F),$$

where the determinant on the left is evaluated by expanding as usual, but being careful to read the composition of operators from left to right, since they do not commute.

This is the formula which generalizes. If F is a function of m variables from V, and $\dim V = m$, define, following Cayley,

$$\Omega(F) = \sum_{\sigma \in \mathfrak{S}_m} \mathrm{sgn}(\sigma) \frac{\partial^m F}{\partial x_{\sigma(1)}^{(1)} \cdot \ldots \cdot \partial x_{\sigma(m)}^{(m)}}; \qquad \text{(F.5)}$$

in symbols, Ω is given by the determinant

$$\begin{vmatrix} \frac{\partial}{\partial x_1^{(1)}} & \frac{\partial}{\partial x_1^{(2)}} & \cdots & \frac{\partial}{\partial x_1^{(m)}} \\ \frac{\partial}{\partial x_2^{(1)}} & & \cdots & \frac{\partial}{\partial x_2^{(m)}} \\ \vdots & & & \vdots \\ \frac{\partial}{\partial x_m^{(1)}} & & \cdots & \frac{\partial}{\partial x_m^{(m)}} \end{vmatrix}.$$

The *Capelli identity* is the formula:

$$\begin{vmatrix} D_{11} + m - 1 & D_{12} & \cdots & D_{1m} \\ D_{21} & D_{22} + m - 2 & \cdots & D_{2m} \\ \vdots & & & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{mm} \end{vmatrix} = [x^{(1)} x^{(2)} \ldots x^{(m)}] \cdot \Omega. \qquad \text{(F.6)}$$

This is an identity of operators acting on functions $F = F(x^{(1)}, \ldots, x^{(m)})$ of m variables, with $m = n = \dim V$, and as always the determinant is expanded with compositions of operators reading from left to right. Note the important corollary: if the number of variables is greater than the dimension, $m > n$, then

$$\begin{vmatrix} D_{11} + m - 1 & D_{12} & \dots & D_{1m} \\ D_{21} & D_{22} + m - 2 & \dots & D_{2m} \\ \vdots & & & \vdots \\ D_{m1} & D_{m2} & \dots & D_{mm} \end{vmatrix} (F) = 0. \tag{F.7}$$

This follows by regarding F as a function on $\mathbb{C}^m$ which is independent of the last $m - n$ coordinates. Since $\Omega(F) = 0$ for such a function, (F.7) follows from (F.6).

We will prove Capelli's identity in §F.3. Now we use it to compute invariants. Let K denote the operator on the left-hand side of these Capelli identities. The expansion of K has a main diagonal term, the product of the diagonal entries $D_{ii} + m - i$, which are scalars on multihomogeneous functions. Note that in any other product of the expansion, the last nondiagonal term which occurs is one of the D_{ij} with $i < j$. Since the diagonal terms commute with the others, we can group the products that precede a given D_{ij} into one operator, so we can write, for $F \in S^{\mathbf{d}}$,

$$K(F) = \rho \cdot F - \sum_{i<j} P_{ij} D_{ij}(F),$$

where $\rho = (d_1 + m - 1) \cdot (d_2 + m - 2) \cdot \ldots \cdot (d_m)$, and each P_{ij} is a linear combination of compositions of various D_{ab}. Capelli's identities say that

$$\rho \cdot F = \sum_{i<j} P_{ij} D_{ij}(F) \qquad \text{if } m > n; \tag{F.8}$$

$$\rho \cdot F = \sum_{i<j} P_{ij} D_{ij}(F) + [x^{(1)} \ldots x^{(m)}] \cdot \Omega(F) \quad \text{if } m = n. \tag{F.9}$$

Just as in the above special case, if F is an invariant of a group G, each $D_{ij}(F)$ is also an invariant in a $S^{\mathbf{d}'}$ where we will know all such invariants by induction. If G is a subgroup of $\mathrm{SL}(V)$, and $m = n$, then $\Omega(F)$ is also an invariant, as follows from the definition or Capelli's identity.

Invariants for $\mathrm{SL}_n\mathbb{C}$.

Let $F \in S^{\mathbf{d}}$ be an invariant of the group $\mathrm{SL}_n\mathbb{C}$. We must show that F can be written as a polynomial in the basic bracket polynomials. In particular, if $m < n$, we must verify that there are no invariants except the constants in $S^0 = \mathbb{C}$. This is a simple consequence of the fact that for a dense open set of m-tuples of vectors—namely, those which are linearly independent—there is an automorphism of $\mathrm{SL}_n\mathbb{C}$ taking them to a fixed m-tuple of independent vectors, say $e_1, \ldots, e_m$. So an invariant function must take the same value on all such m-tuples. By the density, it must be constant.

For $m \geq n$, we proceed by induction as indicated above. All $D_{ij}F$ are known to be invariants (for $i < j$), as is $\Omega(F)$, so these are polynomials in the brackets.

To complete the proof, by Capelli's identities (F.8) and (F.9), it suffices to see that the operators D_{ab} all take brackets to scalar multiples of brackets. This is an obvious calculation: D_{ab} takes a bracket $[x^{(i_1)}\, x^{(i_2)} \dots x^{(i_n)}]$ to zero if b does not appear as one of the superscripts, or to the bracket with the variable $x^{(b)}$ replaced by $x^{(a)}$ if $x^{(b)}$ does occur; the latter is zero if $x^{(a)}$ also occurs and is a bracket otherwise. To avoid repeats, one needs only consider brackets where the superscripts are increasing. This completes the proof of

Proposition F.10. *Polynomial invariants* $F(x^{(1)}, \dots, x^{(m)})$ *of* $\mathrm{SL}_n\mathbb{C}$ *can be written as polynomials in the brackets*

$$[x^{(i_1)}\, x^{(i_2)} \dots x^{(i_n)}], \quad 1 \le i_1 < i_2 < \cdots < i_n \le m.$$

Exercise F.11. Show that the only polynomial invariants of $\mathrm{GL}_n\mathbb{C}$ are the constants.

Invariants for $\mathrm{Sp}_n\mathbb{C}$

Let $r = n/2$, and let Q be the skew form defining the symplectic group $\mathrm{Sp}_n\mathbb{C}$, e.g. $Q(x, y) = \sum_{i=1}^{r} x_i y_{r+i} - x_{r+i} y_i$ in standard coordinates. Note first that the brackets are not needed:

Exercise F.12*. Show that the bracket $[x^{(1)}\, x^{(2)} \dots x^{(n)}]$ is equal to

$$\sum \operatorname{sgn}(\sigma) Q(x^{(\sigma(1))}, x^{\sigma(2)}) \cdot Q(x^{(\sigma(3))}, x^{\sigma(4)}) \cdot \ldots \cdot Q(x^{(\sigma(n-1))}, x^{\sigma(n)}),$$

where the sum is over all permutations σ of $\{1, \dots, n\}$ such that $\sigma(2i-1) < \sigma(2i)$ for $1 \le i \le r$ and $\sigma(i-1) < \sigma(i)$ for $2 \le i \le r$.

Let T_n^m be the assertion that any $\mathrm{Sp}_n\mathbb{C}$-invariant polynomial in m variables from $\mathbb{C}^n$ can be written as a polynomial in the basic polynomials $Q(x^{(i)}, x^{(j)})$. The antilexicographic induction using the Capelli identities is the same as before, and gives the implications

$$T_n^{n-1} \Rightarrow T_n^n \Rightarrow T_n^m \quad \text{for all } m > n.$$

The only variation here is to verify that the operators D_{ab} preserve polynomials in the basic invariants, and $D_{ab} Q(x^{(i)}, x^{(j)})$ is again zero or another basic invariant.

The situation where $m < n$ is a little more complicated than that for the special linear group, however—which is hardly surprising since there are nontrivial invariants for $\mathrm{Sp}_n\mathbb{C}$ in this range. Note that T_n^m implies $T_n^{m'}$ for $m' < m$, so it suffices to prove T_n^{n-1}. This is done by induction on $r = n/2$, i.e., by proving the implication $T_{n-2}^{n-1} \Rightarrow T_n^{n-1}$. To prove this, consider the restriction F' of an invariant polynomial F on $V = \mathbb{C}^n$ to the subspace $V' = \mathbb{C}^{n-2}$ perpendicular to the plane spanned by e_r and e_n. This restriction is an invariant of the group $\mathrm{Sp}_{n-2}\mathbb{C}$. By induction, F' is a polynomial in the basic invariants. Since $Q(x^{(i)}, x^{(j)})$ restricts to the corresponding invariant on V', there is a polynomial in these $Q(x^{(i)}, x^{(j)})$ such that F and this polynomial have the same restriction to V'. Subtracting, it suffices to prove that if an invariant F restricts to zero on V', then F is zero.

We show first that the restriction of F to the larger subspace $W = V' \oplus \mathbb{C}e_r$ must be zero. Fix $y^{(1)}, \ldots, y^{(m)}$ in V', and consider the function of m complex variables.

$$h(t_1, \ldots, t_m) = F(y^{(1)} + t_1 e_r, \ldots, y^{(m)} + t_m e_r).$$

The fact that F is invariant by automorphisms in $\mathrm{Sp}_n\mathbb{C}$ which fix V' and send e_r to $\alpha \cdot e_r$ and e_n to $\alpha^{-1} \cdot e_n$ shows that

$$h(\lambda t_1, \ldots, \lambda t_m) = h(t_1, \ldots, t_m) \quad \text{for all } \lambda \neq 0.$$

Since h is a polynomial, it must be constant, so $h(t_1, \ldots, t_m) = h(0, \ldots, 0) = 0$, as required.

Since F is invariant, it follows that the restriction of F to any hyperplane of the form $g \cdot W$, for any $g \in \mathrm{Sp}_n\mathbb{C}$ is zero. It is not hard to verify that every hyperplane in $\mathbb{C}^n$ has this form. So any $n - 1$ vectors lie in such an hyperplane, and so F is identically zero. This finishes the proof for the symplectic group:

Proposition F.13. *Polynomial invariants $F(x^{(1)}, \ldots, x^{(m)})$ of $\mathrm{Sp}_n\mathbb{C}$ can be written as polynomials in functions*

$$Q(x^{(i)}, x^{(j)}), \quad 1 \leq i < j \leq m.$$

Invariants for $\mathrm{SO}_n\mathbb{C}$

This time brackets may be needed, as well as the functions given by the symmetric form Q, but products of brackets are not required:

Exercise F.14. Prove the identity

$$[x^{(1)}\, x^{(2)} \ldots x^{(n)}] \cdot [y^{(1)}\, y^{(2)} \ldots y^{(n)}] = |Q(x^{(i)}, y^{(j)})|_{1 \leq i,j \leq n}$$

for any variables $x^{(1)}, \ldots, x^{(n)}, y^{(1)}, \ldots, y^{(n)}$.

Let T_n^m be the assertion that any $\mathrm{SO}_n\mathbb{C}$-invariant polynomial in m variables can be written as a polynomial in the brackets and the invariants $Q(x^{(i)}, x^{(j)})$, where we take $Q(x, y) = \sum_{i=1}^n x_i y_i$ to be the form determining the orthogonal group. The proofs of the implications $T_n^{n-1} \Rightarrow T_n^n \Rightarrow T_n^m$ for $m > n$ are exactly as in the preceding cases, and require no further comment. As before, it remains to prove T_n^{n-1}, and, by induction on n, it suffices to prove the implication $T_{n-1}^{n-1} \Rightarrow T_n^{n-1}$.

Let $V' = \mathbb{C}^{n-1}$ be the orthogonal complement to e_n. The restriction F' to V' of an $\mathrm{SO}_n\mathbb{C}$-invariant polynomial F is $\mathrm{SO}_{n-1}\mathbb{C}$-invariant, and by induction we know it is a polynomial in the restrictions of the basic polynomials $Q(x^{(i)}, x^{(j)})$ and in the bracket $[x^{(1)} \dots x^{(n-1)}]$. An apparent snag is met here, however, since this bracket is not the restriction of an invariant on V. By Exercise F.14, we can write

$$F' = A + B\cdot[x^{(1)} \dots x^{(n-1)}],$$

where A and B are polynomials in the Q's alone. In particular, A and B are *even*, i.e., they are invariants of the full orthogonal group $\mathrm{O}_{n-1}\mathbb{C}$. But F' is also even, since any element of $\mathrm{O}_{n-1}\mathbb{C}$ is the restriction of some element in $\mathrm{SO}_n\mathbb{C}$ (mapping e_n to $\pm e_n$). Since the bracket is taken to minus itself by automorphisms of determinant -1, we must have $F' = A$. This means that we can subtract a polynomial in the invariants $Q(x^{(i)}, x^{(j)})$ from F, so we can assume $F' = 0$. Therefore, the restriction of F to any hyperplane of the form $g\cdot V'$, $g \in \mathrm{SO}_n\mathbb{C}$, is zero. But it is easy to verify that $(n-1)$-tuples in such hyperplanes form an open dense subset of all $(n-1)$-tuples in $\mathbb{C}^n$ (the condition is that there be an orthogonal vector e with $Q(e\cdot e) \neq 0$). This proves:

Proposition F.15. *Polynomial invariants $F(x^{(1)}, \dots, x^{(m)})$ of $\mathrm{SO}_n\mathbb{C}$ can be written as polynomials in functions*

$$Q(x^{(i)}, x^{(j)}) \quad \textit{and} \quad [x^{(i_1)} x^{(i_2)} \dots x^{(i_n)}],$$

with $1 \leq i \leq j \leq m$, $1 \leq i_1 < i_2 < \cdots < i_n \leq m$.

Exercise F.16*. Show that the polynomial invariants of $\mathrm{O}_n\mathbb{C}$ can be written as polynomials in the functions $Q(x^{(i)}, x^{(j)})$, $1 \leq i < j \leq m$. Show that *odd* polynomial invariants of $\mathrm{O}_n\mathbb{C}$, i.e., polynomials F which are taken to $\det(g)\cdot F$ by g in $\mathrm{O}_n\mathbb{C}$, can be written as linear combinations of even invariants times brackets.

§F.2. Applications to Symplectic and Orthogonal Groups

We consider the symplectic group $\mathrm{Sp}_n\mathbb{C}$ and the orthogonal group $\mathrm{O}_n\mathbb{C}$ together, letting Q denote the corresponding skew or symmetric form. The results in the first section, applied to the case $\mathbf{d} = (1, \dots, 1)$, say that the invariants in $(V^*)^{\otimes m}$ are all polynomials in the polynomials $Q(x^{(i)}, x^{(j)})$, and by degree considerations m must be even, and they are all linear combinations of products

$$Q(x^{(\sigma(1))}, x^{(\sigma(2))})\cdot Q(x^{(\sigma(3))}, x^{(\sigma(4))})\cdot \ldots \cdot Q(x^{(\sigma(m-1))}, x^{(\sigma(m))}) \qquad \text{(F.17)}$$

for permutations σ of $\{1, \ldots, m\}$ such that $\sigma(2i-1) < \sigma(2i)$ for $1 \le i \le m/2$. Regarding $Q \in V^* \otimes V^*$, these are obtained from the invariant $Q \otimes \cdots \otimes Q$ ($m/2$ times) by permuting the factors. In other words, one pairs off the m components, and inserts Q in the place indicated by each pair.

The form Q gives an isomorphism of V with V^*, which takes v to $Q(v, -)$. Using this we can find all invariants of tensor products $(V^*)^{\otimes k} \otimes (V)^{\otimes l}$, via the isomorphism

$$(V^*)^{\otimes(k+l)} = (V^*)^{\otimes k} \otimes (V^*)^{\otimes l} \cong (V^*)^{\otimes k} \otimes (V)^{\otimes l}.$$

They are linear combinations of the images of the above invariants under this identification. To see what they are, we just need to see what happens to Q under the isomorphisms $V^* \otimes V^* \cong V^* \otimes V$ and $V^* \otimes V^* \cong V \otimes V$:

Exercise F.18. (i) Verify that under the canonical isomorphism

$$V^* \otimes V^* \cong V^* \otimes V = \mathrm{Hom}(V, V) = \mathrm{End}(V)$$

Q maps to the identity endomorphism. (ii) Let ψ be the image of Q under the canonical isomorphism $V^* \otimes V^* \cong V \otimes V$. Verify that

$$\psi = \sum_{i=1}^{r} e_i \otimes e_{r+i} - e_{r+i} \otimes e_i \quad \text{for } G = \mathrm{Sp}_n\mathbb{C},\ n = 2r;$$

$$\psi = \sum_{i=1}^{n} e_i \otimes e_i \quad \text{for } G = \mathrm{O}_n\mathbb{C}.$$

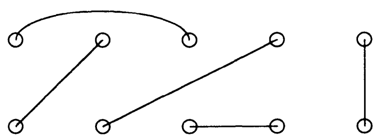
For the applications in Lectures 17 and 19, we need only the case $l = k$, but we want to reinterpret these invariants by way of the canonical isomorphism

$$(V^*)^{\otimes 2d} \cong (V^*)^{\otimes d} \otimes (V)^{\otimes d} \cong \mathrm{Hom}(V^{\otimes d}, V^{\otimes d}) = \mathrm{End}(V^{\otimes d}). \quad \text{(F.19)}$$

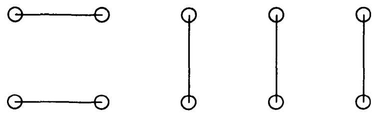
In §§17.3 and 19.5 we defined endomorphisms $\vartheta_I \in \mathrm{End}(V^{\otimes d})$ for each pair I of integers from $\{1, \ldots, d\}$; for I the first pair,

$$\vartheta_I(v_1 \otimes v_2 \otimes v_3 \otimes \cdots \otimes v_d) = Q(v_1, v_2) \cdot \psi \otimes v_3 \otimes \cdots \otimes v_d;$$

the case for general I is a permutation of this. We claim that an invariant in $(V^*)^{\otimes 2d}$ of the form (F.17) is taken by the isomorphism (F.19) to a composition of operators ϑ_I and permutations σ in $\mathfrak{S}_d$. This is simply a matter of unraveling the definitions, which may be simpler to follow pictorially than notationally. The invariant in (F.17) is described by pairing the integers from 1 to $2d$. These pairs are either from the first d, the last d, or one of each. For example, if $d = 5$ the pairings could be as indicated:



for the pairs {1, 3}, {8, 9}, {2, 6}, {4, 7}, {5, 10}. Composing before and after with permutations, this can be changed to



The corresponding endomorphism of $V^{\otimes 5}$ becomes ϑ_I, $I = \{1, 2\}$. The general invariant one gets can be expressed in the form

$$\sigma \circ \vartheta_{I_1} \circ \vartheta_{I_2} \circ \cdots \circ \vartheta_{I_p} \circ \tau,$$

where σ and τ permute the d factors, and the pairs I_j are the first p pairs: $I_j = \{2j - 1, 2j\}$.

Now let A be the subalgebra of the ring $\text{End}(V^{\otimes d})$ generated by all $g \otimes \cdots \otimes g$ for g in the group $G = \text{Sp}_n\mathbb{C}$ (or $\text{O}_n\mathbb{C}$). By the simplicity of the group, we know that A is a semisimple algebra of endomorphisms. We have just computed that the ring B of commutators of A is the ring generated by all permutations in $\mathfrak{S}_d$ and the operators ϑ_I. By the general theory of semisimple algebras, cf. §6.2, A must be the commutator algebra of B. In English, *any endomorphism of* $V^{\otimes d}$ *which commutes with permutations and with the operators* ϑ_I *must be a finite linear combination of operators of the form* $g \otimes \cdots \otimes g$ *for* g *in* G. This is precisely the fact from invariant theory that was used in the text.

We remark that a similar procedure can be used for $\text{SL}_n\mathbb{C}$, but since in this case V and V^* are not isomorphic, to do this one must first do some more work to compute invariants in tensor products of covariant and contravariant factors. The idea is simple enough: use the canonical isomorphism $V \cong \bigwedge^{n-1}(V^*)$ to turn each V factor into several V^* factors. Tracing through the invariants by this procedure is rather complicated, however, and we refer to [Wel, II.8] for details. We did not need this analysis, because it was easy to work the commutator story the other way around, showing that the commutator of $\mathbb{C}[\mathfrak{S}_d]$ is the algebra generated by all $g \otimes \cdots \otimes g$ for g in $\text{SL}_n\mathbb{C}$ (or $\text{GL}_n\mathbb{C}$). This can, in turn, be run backwards:

Exercise F.20*. Use the fact that the the $\text{GL}_n\mathbb{C}$-invariants of $\text{End}(V^{\otimes d})$ are generated by permutations to show that the $\text{GL}_n\mathbb{C}$-invariants of $(V^*)^{\otimes d} \otimes V^{\otimes d}$ are obtained by pairing off the factors and contracting. There are no $\text{GL}_n\mathbb{C}$-invariants in $(V^*)^{\otimes k} \otimes V^{\otimes l}$ if $k \neq l$. For $\text{SL}_n\mathbb{C}$-invariants, one also has determinant factors when $k - l$ is a multiple of the dimension.

We also omit any discussion of the *second fundamental theorems*, which describe the relations among the generators of the rings of invariants (but see the discussions at the ends of Lectures 17 and 19). These results can also be found in [Wel].

§F.3. Proof of Capelli's Identity

The proof is not essentially different from the case $m = 2$, once one has a good notational scheme to keep track of the algebraic manipulations which come about because the basic operators D_{ij} do not commute with each other. A convenient way to do this is as follows. For indices $i_1, j_1, \dots, i_p, j_p$ between 1 and m, define an operator $\Delta_{i_1 j_1}\Delta_{i_2 j_2} \cdots \Delta_{i_p j_p}$ which takes a function F of m variables $x^{(1)}, \dots, x^{(m)}$ to the function

$$\Delta_{i_1 j_1} \cdots \Delta_{i_p j_p}(F) = \sum_{k_1, \dots, k_p = 1}^{n} x_{k_1}^{(i_1)} \cdot \dots \cdot x_{k_p}^{(i_p)} \cdot \frac{\partial^p F}{\partial x_{k_1}^{(j_1)} \cdot \dots \cdot \partial x_{k_p}^{(j_p)}}.$$

For $p = 1$, Δ_{ij} is just the operator D_{ij}, but for $p > 1$, this is *not* the composition of the operators $\Delta_{i_k j_k}$. Note that the order of the terms in the expression $\Delta_{i_1 j_1} \cdots \Delta_{i_p j_p}$ is unimportant.

We can form determinants of $p \times p$ matrices with entries these Δ_{ij}, which act on functions by expanding the determinant as usual, with each of the $p!$ products operating as above. For example, for the $m \times m$ matrix (Δ_{ij}),

$$|\Delta_{ij}|(F) = \sum_{\sigma \in \mathfrak{S}_m} \operatorname{sgn}(\sigma) \cdot \Delta_{1\sigma(1)}\Delta_{2\sigma(2)} \cdot \dots \cdot \Delta_{m\sigma(m)}(F).$$

The matrix (Δ_{ij}) is a product of matrices $(x_k^{(i)}) \cdot (\partial/\partial x_k^{(j)})$, and taking determinants gives the

Lemma F.21. *For* $m = n$, $|\Delta_{ij}|(F) = [x^{(1)} \dots x^{(m)}] \cdot \Omega(F)$.

To prove Capelli's identity (F.6), then, we must prove the following identity of operators on functions $F(x^{(1)}, \dots, x^{(m)})$:

$$\begin{vmatrix} D_{11} + m - 1 & D_{12} & \cdots & D_{1m} \\ D_{21} & D_{22} + m - 2 & \cdots & D_{2m} \\ \vdots & & & \vdots \\ D_{m1} & D_{m2} & \cdots & D_{mm} \end{vmatrix} = \begin{vmatrix} \Delta_{11} & \Delta_{12} & \cdots & \Delta_{1m} \\ \Delta_{21} & \Delta_{22} & \cdots & \Delta_{2m} \\ \vdots & & & \vdots \\ \Delta_{m1} & \Delta_{m2} & \cdots & \Delta_{mm} \end{vmatrix} \qquad \text{(F.22)}$$

This is a formal identity, based on the simple identities:

$$D_{qp} \circ D_{ab} = D_{qp}\Delta_{ab} = \Delta_{qp}\Delta_{ab} \quad \text{if } p \neq a;$$

$$D_{qp} \circ D_{ab} = \Delta_{qp}\Delta_{ab} + D_{qb} \qquad \text{if } p = a.$$

Similarly, if $p \neq a_k$ for all k, then

$$D_{qp} \circ \Delta_{a_1 b_1} \cdots \Delta_{a_r b_r} = \Delta_{qp}\Delta_{a_1 b_1} \cdots \Delta_{a_r b_r}; \qquad \text{(F.23)}$$

while if there is just one k with $p = a_k$, then

$$D_{qp} \circ \Delta_{a_1 b_1} \cdots \Delta_{a_r b_r} = \Delta_{qp}\Delta_{a_1 b_1} \cdots \Delta_{a_r b_r} + \Delta_{a_1 b_1} \cdots \Delta_{q b_k} \cdots \Delta_{a_r b_r} \qquad \text{(F.24)}$$

where in the last term the $\Delta_{q b_k}$ replaces $\Delta_{a_k b_k}$.

We prove (F.22) by showing inductively that all $r \times r$ minors of the two matrices of (F.22) which are taken from the last r columns are equal (as operators on functions F as always). This is obvious when $r = 1$. We suppose it has been proved for $r = m - p$, and show it for $r + 1$. By induction, we may replace the last r columns of the matrix on the left by the last r columns of the matrix on the right. The difference of a minor on the left and the corresponding minor on the right will then be a maximal minor of the matrix

$$\begin{vmatrix} D_{1p} - \Delta_{1p} & \Delta_{1p+1} & \cdots & \Delta_{1m} \\ D_{2p} - \Delta_{2p} & \Delta_{2p+1} & \cdots & \Delta_{2m} \\ \vdots & & & \vdots \\ D_{pp} - \Delta_{pp} + r & \Delta_{pp+1} & \cdots & \Delta_{pm} \\ \vdots & & & \vdots \\ D_{mp} - \Delta_{mp} & \Delta_{mp+1} & \cdots & \Delta_{mm} \end{vmatrix},$$

so we must show that all maximal minors of this matrix are zero. Suppose the minor chosen is that using the q_ith rows, for $1 \le q_0 < q_1 < \cdots < q_r \le m$. Expanding along the left column, this determinant is

$$E_0 M_0 - E_1 M_1 + E_2 M_2 - \cdots + (-1)^r E_r M_r, \tag{F.25}$$

where $E_k = D_{q_k p} - \Delta_{q_k p}$ if $q_k \neq p$, and $E_k = D_{pp} - \Delta_{pp} + r$ if $q_k = p$, and M_k is the corresponding cofactor ($r \times r$) determinant:

$$\sum_{\sigma \in \mathfrak{S}_r} \operatorname{sgn}(\sigma) \Delta_{q_0 p+\sigma(1)} \cdots \Delta_{q_{k-1} p+\sigma(k)} \Delta_{q_{k+1} p+\sigma(k+1)} \cdots \Delta_{q_r p+\sigma(r)}. \tag{F.26}$$

To show that (F.25) is zero, there are two cases. In the first case, the pth row is not included in the minor, i.e., $q_i \neq p$ for all i. In this case each term $E_i M_i$ is zero, since $E_i = D_{q_i p} - \Delta_{q_i p}$, and all the products in the expansion of M_i are of the form $\Delta_{a_1 b_1} \ldots \Delta_{a_r b_r}$ with all $a_i \neq p$, and the assertion follows from (F.23).

In the second case, the pth row is included, i.e., $q_k = p$ for some k. As in the first case, $(D_{pp} - \Delta_{pp}) M_k = 0$, and since $E_k = D_{pp} - \Delta_{pp} + r$, we have

$$E_k M_k = r \cdot M_k.$$

We claim that each of the other terms $E_i M_i$, for $i \neq k$, is equal to $(-1)^{k-i+1} M_k$, from which it follows that the alternating sum in (F.25) is zero. When M_i is written out as in (F.26), and it is multiplied by $E_i = D_{q_i p} - \Delta_{q_i p}$, an application of (F.24) shows that one gets the same determinant as (F.26), but expanded with the q_ith row moved between the q_{k-1}th and the q_{k+1}th rows. This transposition of rows accounts for the sign $(-1)^{k-i+1}$, yielding $E_i M_i = (-1)^{k-i+1} M_k$, as required. □

Exercise F.27. Find a $GL(V)$-linear surjection from $S^{d_1} \otimes \cdots \otimes S^{d_n}$ onto $\bigwedge^n V^* \otimes S^{d_1 - 1} \otimes \cdots \otimes S^{d_n - 1}$ that realizes the map $F \mapsto [x^{(1)} \ldots x^{(n)}] \cdot \Omega(F)$.

Hints, Answers, and References

Note: Usually answers or references are given only for more theoretical exercises, or those which may be referred to elsewhere.

Lecture 1

(1.3) The hypotheses ensure that $\bigwedge^n V$ is trivial, and the bilinear map $\bigwedge^k V \otimes \bigwedge^{n-k} V \to \bigwedge^n V = \mathbb{C}$ is a perfect pairing, i.e., it makes each space the dual of the other, cf. §B.3.

(1.4) For (b), take the function α to the function α', where $\alpha'(g) = \alpha(g^{-1})$.

(1.13) Yes. See Exercise 6.18.

(1.14) If H is a Hermitian inner product on V, let $\tilde{H}$: $V \to V^*$ be the conjugate linear map given by $v \mapsto H(v, \cdot)$. If H' is another, the composite $(\tilde{H}')^{-1} \circ \tilde{H}$ is linear, and a G-homomorphism if H and H' are G-invariant. Apply Schur's lemma.

Lecture 2

(2.3) For a general formula expressing complete symmetric polynomials and elementary symmetric polynomials in terms of sums of powers, see Exercise A.32(vi).

(2.4) Look at the induced action on $\bigwedge^k V$.

(2.7) $V^{\otimes n} = U^{\oplus a} \oplus U'^{\oplus b} \oplus V^{\oplus c}$, with $a = b = \frac{1}{3}(2^{n-1} + (-1)^n)$, and $c = \frac{1}{3}(2^n + (-1)^{n-1})$.

(2.25) Answers: (i) $U \oplus V \oplus U' \oplus V'$; (ii) $U \oplus V^{\oplus 2} \oplus V' \oplus W$.

(2.29) The regular representation will do.

(2.33) For (c) use characters or the isomorphism

$$\operatorname{Hom}_G(V \otimes W, U) \cong \operatorname{Hom}_G(W, V^* \otimes U).$$

(2.34) Schur's lemma applies to L.

(2.35) Apply the preceding exercise, with L_0 given by a matrix of indeterminates. For details, see [Se2, §2.2].

(2.36) Show that $(\chi, \chi) = 1$, and compute the sum of the squares of these representations. Reference: [Se2, §3.2].

(2.37) If φ is the character of an irreducible representation, and χ is the character of V, let $a_n = (\varphi, \chi^n)$, and consider the power series

$$\sum_{n=0}^{\infty} a_n t^n = \frac{1}{|G|} \sum_{n=0}^{\infty} \sum_C |C| \overline{\varphi(C)} \chi(C)^n t^n = \frac{1}{|G|} \sum_C \frac{|C| \overline{\varphi(C)}}{(1 - \chi(C)t)}.$$

Here C runs over conjugacy classes. Since $\chi(C) = \dim(V)$ only for $C = [e]$, the right-hand side is a nontrivial rational function; in particular a_n cannot be zero for all positive n.

(2.38) This is another theorem of Burnside. If C is a conjugacy class in G, $\varphi = \sum_{g \in C} g\colon V \to V$ is a G-map, so multiplication by a scalar λ_C, and $\lambda_C \cdot \dim V = \operatorname{Trace}(\varphi) = |C| \cdot \chi_V(C)$. The λ_C are algebraic integers, since the elements $\sum_{g \in C} e_g$, as C varies over the conjugacy classes, generate the center of the group ring $\mathbb{Z}[G]$, which is a finitely generated abelian group. Now

$$\sum_C |C| \cdot \overline{\chi_V(C)} \chi_V(C) = |G|,$$

so $|G|/\dim V = \sum_C \lambda_C \cdot \overline{\chi_V(C)}$ is an algebraic integer. In fact, the dimension of V divides the index of the center of G, cf. [Se2, p. 53].

(2.39) In case the character χ is $\mathbb{Z}$-valued, the equation $\sum |\chi(g)|^2 = |G|$ shows that $|G|$ is the sum of $|G|$ non-negative integers, one of which, $|\chi(e)|^2$, is greater than 1, so at least one must be 0. In general, the values of χ are algebraic integers, since they are sums of roots of unity. Let $\chi_1, \ldots \chi_m$ be the characters obtained from χ by the action of the Galois group $\operatorname{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ (or $\operatorname{Gal}(\mathbb{C}/\mathbb{Q})$) on χ; these characters are also characters of irreducible representations of G. Now if $\chi(g) \neq 0$, then $\prod_i \chi_i(g)$ is a nonzero integer, so $|\prod_i \chi_i(g)|^2 \geq 1$. Since the arithmetic mean is at least the geometric mean, $\sum_i |\chi_i(g)|^2 \geq m$. Therefore,

$$m|G| = \sum_{i=1}^{m} \sum_{g \in G} |\chi_i(g)|^2 \geq m|G|,$$

and we must have equality for every $g \in G$. In particular, if d is the degree of the representation, $md^2 = \sum_i |\chi_i(e)|^2 = m$, so $d = 1$.

Lecture 3

(3.5) Use the fact that $g = (12345)$ is conjugate to its inverse, so $\chi(g) = \chi(g^{-1}) = \overline{\chi(g)}$ is real.

(3.25) See §5.1.

(3.26) If $H \subset G$ is the subgroup of order 7, there are three one-dimensional representations from G/H, and two three-dimensional representations induced from H. For generalizations, see [Se2, §8.2].

(3.30) W is embedded in the space of W-valued functions on G by sending $w \in W$ to the function which takes $h \in H$ to $h \cdot w$ and all other cosets to zero. Note that if $\{g_\sigma\}$ is a set of coset representatives, the map $f \mapsto \sum g_\sigma \otimes f(g_\sigma^{-1})$ gives an isomorphism from $\mathrm{Hom}_H(\mathbb{C}G, W)$ to $\mathbb{C}G \otimes_{\mathbb{C}H} W$.

(3.32) For (b), identify the right-hand side with the trace of an endomorphism of $\mathbb{C}G$. For (c), take φ to be the characteristic function of an element g and apply (b).

(3.33) F is the determinant of left multiplication by the element $a = \sum x_g e_g \in \mathbb{C}G$ on the regular representation, and F_ρ is the determinant of left multiplication by a on the irreducible $\mathbb{C}G$-module V_ρ corresponding to ρ. The factorization of F follows from the decomposition of the regular representation. The irreducibility of F_ρ follows from the irreducibility of a matrix whose entries are indeterminates, using Proposition 3.29. Fixing g in G, set the variables $x_e = 1$ and $x_h = 0$ for $h \neq g$; the coefficient of x_g in the determinant of left multiplication by $1 + x_g e_g$ on V_ρ is $\chi_\rho(g)$.

(3.34) See Exercises 3.8 and 3.9.

(3.38) V can be replaced by V^*; $V \otimes V = \mathrm{Sym}^2 V \oplus \wedge^2 V$ contains at most one copy of the trivial representation. If $\mathrm{Sym}^2 V$ contains the trivial representation, then

$$|G| = \sum_{g \in G} \chi_{\mathrm{Sym}^2 V}(g) = \tfrac{1}{2}\left(\sum \chi_V(g)^2 + \sum \chi_V(g^2)\right).$$

Otherwise, the right-hand side is zero; similarly for $\wedge^2 V$. Note that if χ_V is real, then $\sum \chi_V(g)^2 = |G|$.

(3.41) Reference: [Se2, §13.2].

(3.42) Reference: [Ja-Ke, p. 12].

(3.43) Consider the endomorphism $J \otimes J$ of $V \otimes W$.

(3.44) For $G = \mathbb{Z}/3$, the rank of $R_{\mathbb{R}}(G)$ is 2, whereas that of $R(G)$ is 3.

(3.45) See [Se2, §12] for details.

Lecture 4

(4.4) Right multiplication by a gives a map $Aab \to Aba$, and right multiplication by b gives a map back. The composites are multiplications by nonzero scalars. More generally, if $A = \mathbb{C}G$ is a group algebra, call an element $a = \sum a_g e_g$ *Hermitian* if $\hat{a} = \bar{a}$, i.e., $a_{g^{-1}} = \overline{a_g}$. If a and b are idempotents which are Hermitian, then $Aab \cong Aba$.

(4.6) A basis for $V_{(d-1,1)} = \mathbb{C}\mathfrak{S}_d \cdot c_\lambda$ is $v_2, \ldots, v_d$, where

$$v_j = \sum_{g(d)=j} e_g - \sum_{h(1)=j} e_h.$$

Note that $v_d = c_\lambda$, $v_1 + \cdots + v_d = 0$, and $g \cdot v_j = v_i$ if $g(j) = i$. A basis for $V \subset \mathbb{C}^d$ is v_2, $\ldots$, v_d, where $v_j = e_j - e_{j-1}$. For the case $s > 1$, use (4.10) or see (4.43).

(4.13) Note that the hook lengths of the boxes in the first column are the numbers l_1, $\ldots$, l_k. Induct from the diagram obtained by omitting the first column.

(4.14) Induct as in the preceding exercise by removing the first column, considering separately the cases when the remaining diagram is one of the exceptions.

(4.15) Frobenius [Fro1] gives these and analogous formulas for $\lambda = (d-3, 3)$, $(d-3, 1, 1, 1)$, $(d-4, 4), \ldots$.

(4.16) Using Frobenius's formula, the coefficient of $x_1^{l_1} \cdot \ldots \cdot x_k^{l_k}$ in $\Delta \cdot (x_1^d + \cdots + x_k^d)$ can be nonzero only if $l_1 = d$, so λ has the prescribed form; the coefficient of $x_2^{k-1} x_3^{k-2} \cdot \ldots \cdot x_k$ in $\Delta(0, x_2, \ldots, x_k)$ is $(-1)^{k-1}$.

(4.19) See Exercise 4.51 for a general procedure for decomposing tensor products.

(4.20) Use Frobenius's formula as in Exercise 4.16 to show that $\chi_\lambda(g) = (-1)^{k-1}\chi_\mu(h)$, where $\mu = (\lambda_2 - 1, \lambda_3 - 1, \ldots, \lambda_k - 1)$ and $h \in \mathfrak{S}_{d-q_1}$ is the product of cycles of lengths $q_2, \ldots, q_r$.

(4.24) If $\lambda < \mu$ use the anti-involution $\hat{}$ of A induced by the map $g \mapsto g^{-1}$, $g \in \mathfrak{S}_d$, noting that $\hat{c}_\lambda = (a_\lambda b_\lambda)\hat{} = \hat{b}_\lambda \hat{a}_\lambda = b_\lambda a_\lambda$, so $(c_\lambda \cdot x \cdot c_\mu)\hat{} = \hat{c}_\mu \cdot \hat{x} \cdot \hat{c}_\lambda = b_\mu \cdot (a_\mu \cdot \hat{x} \cdot b_\lambda) \cdot a_\lambda = 0$.

(4.40) Note that the ψ_λ's are related to the χ_λ's by the same equations as the symmetric polynomials H_λ's to the Schur polynomials S_λ's, cf. (A.9) in the appendix. The equation (A.5) for the S_λ's in terms of the H_λ's therefore implies the determinantal formula.

(4.43) Use Frobenius reciprocity and (4.42) to prove the general formula. To prove that $V_{(d-s,1,\ldots,1)} \cong \wedge^s V$, argue by induction on d. Note that the restriction of $\wedge^s V$ splits into a sum of two exterior powers of the standard representation, and from anything but a hook one can remove at least three boxes.

(4.44) The induced representation of V_λ by the inclusion of $\mathfrak{S}_d$ in $\mathfrak{S}_{d+m}$ is $V_\lambda \circ V_{(m)}$. Use the transitivity of induction, Exercise 3.16(b).

(4.45) For (a), see [Jam, pp. 79–83]. For (b), using (4.33), the coefficient of X^a in $(x_1^m + \cdots + x_k^m) \cdot P^{(\mathbf{i})}$ is the sum of the coefficients of $X^a x_i^{-m}$ in $P^{(\mathbf{i})}$, summing over those i for which $a_i \geq m$. Use the determinantal formula to write $\chi_\lambda(g)$ as a sum $\Sigma \pm \chi_\mu(h)$, and show that the μ which occur are those obtained by removing skew hooks. Reference: [Boe, pp. 192–196].

(4.46) See Exercise A.11. In fact, this condition is equivalent to the condition that $K_{\rho\lambda} \leq K_{\rho\mu}$ for all ρ, or to the condition that U_λ is isomorphic to $U_\mu \oplus W$, for some representation W, cf. [L-V].

(4.47) References: For the first construction see [Jam], [Ja-Ke]; for the second, see [Pe2].

(4.48) There are several ways to do this: (i) Use the methods of this lecture to show that the value of the character of U'_λ on the class $C_{\mathbf{i}}$ is $[\vartheta(P^{(\mathbf{i})})]_{\lambda'}$, where ϑ is the involution defined in Exercise A.32. Then apply Lemma A.26. (ii) Show that $U'_\lambda \otimes U'$ is isomorphic to $U_{\lambda'}$ and use Corollary 4.39. (iii) Use Exercise 4.40 or 4.44.

(4.49) Use Exercise A.32(v).

(4.51) (a) Note that $\chi_\lambda = \sum_{\mathbf{i}} \omega_\lambda(\mathbf{i})\xi_{(\mathbf{i})}$, and $\xi_{(\mathbf{i})} = (1/z(\mathbf{i}))\sum_\nu \omega_\nu(\mathbf{i})\chi_\nu$, where $\xi_{(\mathbf{i})}$ is the characteristic function of the conjugacy class $C_{(\mathbf{i})}$. Therefore,

$$\chi_\lambda \chi_\mu = \sum_{\mathbf{i}} \omega_\lambda(\mathbf{i})\omega_\mu(\mathbf{i})\xi_{(\mathbf{i})},$$

from which the required formula follows. For other procedures and tables for small d see [Ja-Ke], [Co], and [Ham].

(b) $V_\lambda \otimes V_{(d)} = V_\lambda$, and $V_\lambda \otimes V_{(1,\ldots,1)} = V_{\lambda'}$, which prove the corresponding results for $C_{\lambda(d)\mu}$ and $C_{\lambda(1,\ldots,1)\mu}$. Use (a) to permute the subscripts.

(4.52) For (a), the described map from Λ to R is surjective by the determinantal formula of Exercise 4.40; it is an isomorphism since R_n and Λ_n are free of the same rank. For (f), note that $P^{(\mathbf{i})}$ corresponds to the character $\sum_\lambda \chi_\lambda(C_{(\mathbf{i})})\chi_\lambda$, which by Exercise 2.21 is the class function which is zero outside the conjugacy class $C_{(\mathbf{i})}$, and whose value on $C_{(\mathbf{i})}$ is $z(\mathbf{i})$.

For more on this correspondence, see [Bu], [Di2], [Mac]. In [Kn] a λ-ring structure on this ring is related to representation theory. In [Liu] this Hopf algebra is used to derive many of the facts about representations of $\mathfrak{S}_d$ from scratch. In [Ze] a similar approach is also used for representations of $\mathrm{GL}_n(\mathbb{F}_q)$.

More about representations of the symmetric groups can also be found in [Foa] and [J-L].

Lecture 5

(5.2) Consider the class functions on H which are invariant by conjugation by an element not in H.

(5.4) *Step* 1. (i) Inverses of elements of c' are conjugate to elements of c' if m is even, and to elements of c'' if m is odd; $\chi(g^{-1}) = \overline{\chi(g)}$. (ii) (ϑ, ϑ) is

$$\frac{2}{d!}(\#c' \cdot |u - v|^2 + \#c'' \cdot |v - u|^2) = \frac{2}{d!}\frac{d!}{q_1 \cdot \ldots \cdot q_r}|u - v|^2.$$

(iii) If λ corresponded to $p \neq q$, the values of χ'_λ and χ''_λ on the corresponding conjugacy classes $c'(p)$ and $c''(p)$ would be the same number, say w, and Exercise 4.20 implies that $2w = \pm 1$. Since w is an algebraic integer, this is impossible. Therefore, λ corresponds to q, and now from Exercise 4.20 we get the additional equation $u + v = (-1)^m$.

Step 2. (ii) Information about the characters χ' and χ'' of X' and X'' is easily determined from Exercise 3.19, and the fact that the characters of the factors are known by induction. In particular, since $c'(q)$ and $c''(q)$ each decomposes into two conjugacy classes in H, we have

$$\chi'(c'(q)) = \frac{\varepsilon_1 + \sqrt{\varepsilon_1 q_1}}{2}\cdot\frac{\varepsilon' + \sqrt{\varepsilon' q'}}{2} + \frac{\varepsilon_1 - \sqrt{\varepsilon_1 q_1}}{2}\cdot\frac{\varepsilon' - \sqrt{\varepsilon' q'}}{2}$$
$$= \frac{\varepsilon + \sqrt{\varepsilon q_1 \cdot \ldots \cdot q_r}}{2},$$

where $\varepsilon_1 = (-1)^{(q_1 - 1)/2}$, $\varepsilon' = (-1)^{(d - q_1 - r + 1)/2}$, $\varepsilon = \varepsilon_1 \cdot \varepsilon'$, and $q' = q_2 \cdot \ldots \cdot q_r$; and similarly for the other values. (iv) The character of Y takes equal values on each pair of conjugate classes. (Reference: [Fro2], [Boe]).

(5.5) Reference: [Ja-Ke].

(5.9) If N is a normal subgroup properly between $\{\pm 1\}$ and $\mathrm{SL}_2(\mathbb{F}_q)$, one of the nontrivial characters χ must take the value $\chi(1)$ identically on N.

(5.11) Reference: [Ste1].

Lecture 6

(6.4) Compare (1) of the theorem with formulas (4.11) and (4.12). For a procedure to construct a basis of $\mathbb{S}_\lambda V$, see Exercise 6.28.

(6.10) By (4.41), there is an isomorphism of $\mathbb{C}\mathfrak{S}_{d+m}$-modules:

$$\mathbb{C}\mathfrak{S}_{d+m} \otimes_{\mathbb{C}(\mathfrak{S}_d \times \mathfrak{S}_m)} (V_\lambda \boxtimes V_\mu) \cong \bigoplus_\nu N_{\lambda\mu\nu} V_\nu.$$

Tensoring on the left with the right $\mathbb{C}\mathfrak{S}_{d+m}$-module $V^{\otimes(d+m)} = V^{\otimes d} \otimes_{\mathbb{C}} V^{\otimes m}$, and noting that $\mathbb{C}(\mathfrak{S}_d \times \mathfrak{S}_m) = \mathbb{C}\mathfrak{S}_d \otimes \mathbb{C}\mathfrak{S}_m$,

$$(V^{\otimes d} \otimes_{\mathbb{C}} V^{\otimes m}) \otimes_{\mathbb{C}\mathfrak{S}_d \otimes \mathbb{C}\mathfrak{S}_m} (V_\lambda \otimes V_\mu) \cong \bigoplus_\nu N_{\lambda\mu\nu} \mathbb{S}_\nu V.$$

(This also uses the general fact: if $A \to B$ is a ring homomorphism, N a left A-module, and M a right B-module, then $M \otimes_B (B \otimes_A N) \cong M \otimes_A N$.) The left-hand side of the displayed equation is

$$(V^{\otimes d} \otimes_{\mathbb{C}\mathfrak{S}_d} V_\lambda) \otimes_{\mathbb{C}} (V^{\otimes m} \otimes_{\mathbb{C}\mathfrak{S}_m} V_\mu) \cong \mathbb{S}_\lambda(V) \otimes \mathbb{S}_\mu(V),$$

which concludes the proof.

(6.11) (a) The key observation is that

$$(V \oplus W)^{\otimes d} = \bigoplus (V^{\otimes a} \otimes W^{\otimes b}) \otimes_{\mathbb{C}(\mathfrak{S}_a \times \mathfrak{S}_b)} \mathbb{C}(\mathfrak{S}_d),$$

the sum over all a, b with $a + b = d$. Tensoring this on the right with the $\mathbb{C}(\mathfrak{S}_d)$-module V_ν one gets

$$(V \oplus W)^{\otimes d} = \bigoplus (V^{\otimes a} \otimes W^{\otimes b}) \otimes_{\mathbb{C}(\mathfrak{S}_a \times \mathfrak{S}_b)} \mathrm{Res}_{a,b} V_\nu,$$

where $\mathrm{Res}_{a,b}$ denotes the restriction to $\mathfrak{S}_a \times \mathfrak{S}_b$. Then use Exercise 4.43 to decompose this restriction.

(b) By Frobenius reciprocity, the representation induced by V_ν via the diagonal embedding of $\mathfrak{S}_d$ in $\mathfrak{S}_d \times \mathfrak{S}_d$ is $\bigoplus C_{\lambda\mu\nu} V_\lambda \boxtimes V_\mu$. With $A = \mathbb{C}\mathfrak{S}_d$, this says

$$(A \otimes A) \otimes_A Ac_\nu = \bigoplus C_{\lambda\mu\nu}(Ac_\lambda \otimes Ac_\mu).$$

Tensor this with the right $(A \otimes A)$-module $(V \otimes W)^{\otimes d} = V^{\otimes d} \otimes W^{\otimes d}$. The special case follow from Exercise 4.51(b).

(6.13) Use Exercise A.32(iv), or write the left side as $V^{\otimes d} \otimes A \cdot b_\lambda$ and use Exercise 4.48.

(6.14) These come from the realizations of the representation $V_\lambda = Ac_\lambda$ as the image of the maps $Ab_\lambda \to Aa_\lambda$ given by right multiplication by a_λ, and similarly $Aa_\lambda \to Ab_\lambda$ by right multiplication by b_λ.

(6.15) It is clear that if one allows T to vary over all tableaux with strictly increasing columns but no conditions on the rows, then the corresponding v_T span the first space $\bigotimes_i (\wedge^{\mu_i} V)$; to show that the v_T for T semistandard span the image the key point is to show how to interchange elements in successive rows. Once it is checked that the elements span, the independence can be deduced from the fact that the number of semistandard tableaux is the same as the dimension. For a direct proof of both spanning and independence, see [A-B-W]—but note that their partitions are all the conjugates of ours. See also Proposition 15.55.

(6.16) Use Exercise 6.14 to realize each $\mathbb{S}_\lambda V$ which occurs as the image in $V^{\otimes d} \otimes V^{\otimes d}$ of a symmetrizing map, and check whether this image is invariant or anti-invariant by the map which permutes the two factors.

(6.17) (a) Identifying the dm elements on which $\mathfrak{S}_{dm}$ acts with the set of pairs $\{(i, j) | 1 \le i \le d, 1 \le j \le m\}$ determines embeddings of the groups $\mathfrak{S}_d \times \cdots \times \mathfrak{S}_d$ (m factors) and $\mathfrak{S}_m$ in $\mathfrak{S}_{dm}$. Let

$$c' = c_\lambda \otimes \cdots \otimes c_\lambda \in \mathbb{C}\mathfrak{S}_d \otimes \cdots \otimes \mathbb{C}\mathfrak{S}_d = \mathbb{C}(\mathfrak{S}_d \times \cdots \times \mathfrak{S}_d) \subset \mathbb{C}\mathfrak{S}_{dm},$$

$$c'' = c_\mu \in \mathbb{C}\mathfrak{S}_m \subset \mathbb{C}\mathfrak{S}_{dm}.$$

Then $c = c' \cdot c''$ is the required element of $\mathbb{C}\mathfrak{S}_{dm}$. For a combinatorial description of plethysm see [Mac, §I.8].

(b) The answers are

$$\mathrm{Sym}^2(\mathbb{S}_{(2,2)}V) = \mathbb{S}_{(4,4)}V \oplus \mathbb{S}_{(4,2,2)}V \oplus \mathbb{S}_{(3,3,1,1)}V \oplus \mathbb{S}_{(2,2,2,2)}V;$$

$$\Lambda^2(\mathbb{S}_{(2,2)}V) = \mathbb{S}_{(4,3,2)}V \oplus \mathbb{S}_{(3,2,2,1)}V.$$

Reference: [Lit2, p. 278].

(6.18) Their characters are the same. In fact, if x and y are eigenvalues of an endomorphism of V, the trace on the left-hand side is $\sum f(k) x^k y^{pq-k}$, where $f(k)$ is the number of partitions of k into at most p integers each at most q. This number is symmetric in p and q, by conjugating partitions.

(6.19) The facts about skew Schur polynomials are straightforward generalizations of corresponding facts for regular Schur polynomials given in Appendix A; proofs of (i)–(iv) can be found in [Mac]. To see that the two descriptions of $V_{\lambda/\mu}$ agree see the hint for Exercise 4.4(a). Skew Schur functors are discussed in [A-B-W], where the construction of a basis is given; from this the character formula (viii) follows. Then (iv) implies (v) and (ix).

(6.20) References, with proofs of similar statements in arbitrary characteristic (where the results, however, are weaker), are [Pe1] and [Jam].

(6.21) References: [A-B-W] and [P-W].

(6.29) A reference for the general theory of semisimple algebras and its applications to group theory is [C-R, §26].

Lecture 7

(7.1) One way to show that a symplectic transformation has determinant 1, cf. [Di1], is to show that the group $\mathrm{Sp}_{2n}\mathbb{C}$ is generated by those which fix a hyperplane, i.e., transformations of the form $v \mapsto v + \lambda Q(v, u)u$ for some vector u and scalar λ. Another, cf. Exercise F.12, is to write the determinant as a polynomial expression in terms of the form Q.

(7.2) Consider the action on the quadric $Q(v, v) = 1$.

(7.11) For any y, the image of the map $x \mapsto xyx^{-1}y^{-1}$ is discrete only if y is central.

(7.13) $\mathrm{PGL}_n\mathbb{C}$ acts by conjugation on $n \times n$ matrices.

Lecture 8

(8.10) (b) $\mathrm{ad}[X, Y](Z) = [[X, Y], Z]$, and $[\mathrm{ad}\, X, \mathrm{ad}\, Y](Z) = (\mathrm{ad}\, X \circ \mathrm{ad}\, Y - \mathrm{ad}\, Y \circ \mathrm{ad}\, X)(Z) = [X, [Y, Z]] - [Y, [X, Z]]$.

(8.16) The kernel of Ad is the center $Z(G)$, cf. Exercise 7.11.

(8.17) Use statement (ii), noting that W is G-invariant if it is $\tilde{G}$-invariant, $\tilde{G}$ the universal covering of G.

(8.24) With A, B, C, D $n \times n$ matrices,

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} : {}^tAC = {}^tCA, {}^tBD = {}^tDB, {}^tAD - {}^tCB = \mathrm{I}\right\}.$$

$$\mathfrak{sp}_{2n}\mathbb{R} = \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| {}^tB = B, {}^tC = C, {}^tA = -D\right\}.$$

(8.28) The automorphisms of $G = \tilde{G}/C$ are the automorphisms of $\tilde{G}$ which preserve C.

(8.29) The point is that the commutator of two vector fields is again a vector field, which can be checked in local coordinates.

(8.35) Both signs are plus.

(8.38) Reference: [Ho1].

(8.42) For $h \in H$, $H_0 \cdot h$ gives a coordinate neighborhood of h. For another approach to Proposition 8.41, with more details, see [Hel, §II.2].

(8.43) For an example, take any simply connected group which contains a torus of dimension greater than one, say SU(3), and take an irrational line in the torus.

Lecture 9

(9.7) If H is an abelian subgroup of G, and the claim holds for G/H, show that it holds for G. Or, if G is realized as a group of nilpotent matrices, apply Campbell–Hausdorff.

(9.10) If each $\mathrm{ad}(X)$ is nilpotent, the theorem gives a flag $\mathfrak{g} = V_0 \supset V_1 \supset \cdots \supset V_k = 0$, with $[\mathfrak{g}, V_i] \subset V_{i+1}$, from which it follows that $\mathcal{D}_i\mathfrak{g} \subset V_i$.

(9.21) If $\mathfrak{g}$ had an abelian ideal $\mathfrak{a}$, semisimplicity of the adjoint representation would mean that there is a surjection $\mathfrak{g} \to \mathfrak{a}$ of Lie algebras. But an abelian Lie algebra has lots of representations that are not semisimple.

(9.24) For the last statement, note that the adjoint representation is semisimple. Or see Corollary C.11.

(9.25) Reference: [Bour, I] for this (as well as for details for many other statements in Lecture 9).

(9.27) the adjoint representation is semisimple.

Lecture 10

(10.1) Any holomorphic map from E to $\mathbb{C}$ must be constant.

(10.2) An isomorphism $G_n \cong G_m$ would lift to a map $G \to G$; show that this map would have to be an isomorphism.

(10.4) By hypothesis, the Lie algebra $\mathfrak{g}$ of G has an ideal $\mathfrak{h}$ with abelian quotient; use the corresponding exact sequence of groups, with the corresponding long exact homotopy sequence (cf. §23.1), and an induction on the dimension of G.

Lecture 11

(11.11) Verify the combinatorial formula

$$\left(\sum_{i=0}^{a} x^{a-2i}\right)\left(\sum_{j=0}^{b} x^{b-2j}\right) = \sum_{k=0}^{a}\left(\sum_{l=0}^{a+b-2k} x^{a+b-2k-2l}\right).$$

Reference: [B-tD, p. 87]

(11.19) Given two points on C there is a 2-dimensional vector space of quadrics containing C and the chord between the points.

(11.20) Answer: it is the subspace of the space of quadrics spanned by the squares of the osculating planes to the twisted cubic curve.

(11.23) Answer: the cones over the curve, with vertex a varying point in $\mathbb{P}^3$.

(11.25) Look at the chordal variety of the rational normal curve in $\mathbb{P}^4$.

(11.32) The sum for $\alpha \geq k$ corresponds to the quadrics containing the osculating $(k-1)$-planes to the curve.

(11.34) See Exercise 6.18.

(11.35) Reference: [Mur1, §15].

Lecture 13

(13.3) For V standard, $\mathbb{S}_{(a+b,b)}V \cong \Gamma_{a,b}$. See §15.3 for details.

(13.8) If $a, b > 0$, $V \otimes \Gamma_{a,b} = \Gamma_{a+1,b} \oplus \Gamma_{a-1,b+1} \oplus \Gamma_{a,b-1}$, cf. §15.3.

(13.20) Warning: writing out the eigenvalue diagram and performing the algorithm above is probably not the way to do this.

(13.22) The tangent planes to the Veronese surface should span a subrepresentation.

(13.24) See §23.3 for a general description of these closed orbits.

More applications of representation theory to geometry can be found in [Don] and [Gre].

Lecture 14

(14.15) The fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ is proved in Claim 21.19.

(14.33) See the proof of Proposition 14.31.

(14.34) If $\mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{g}_\alpha \neq 0$, then $\mathrm{Rad}(\mathfrak{g}) \supset \mathfrak{s}_\alpha \cong \mathfrak{sl}_2$, which is not solvable. If $\mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{h} \ni H$, and $\alpha(H) \neq 0$, then $\mathfrak{g}_\alpha = [H, \mathfrak{g}_\alpha] \subset \mathrm{Rad}(\mathfrak{g})$. Use the fact that $[\mathfrak{h}, \mathrm{Rad}(\mathfrak{g})] \subset \mathrm{Rad}(\mathfrak{g})$ to conclude that $\mathrm{Rad}(\mathfrak{g}) = \mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{h} + \sum \mathrm{Rad}(\mathfrak{g}) \cap \mathfrak{g}_\alpha = 0$. For a stronger theorem, see [Va, §4.4].

(14.35) If $\mathfrak{b}' \supset \mathfrak{b}$, then $\mathfrak{b}' \supset \mathfrak{h}$, so $\mathfrak{b}'$ is a direct sum of $\mathfrak{h}$ and some root spaces $\mathfrak{g}_\alpha$ for $\alpha \in T$, $T \supsetneqq R^+$. Then T contains some $-\alpha$ together with α, so $\mathfrak{b}' \supset \mathfrak{s}_\alpha \cong \mathfrak{sl}_2$, which is not solvable.

(14.36) For $\mathfrak{sl}_m\mathbb{C}$, $B(X, Y) = 2m\,\mathrm{Tr}(X \circ Y)$. For $\mathfrak{so}_m\mathbb{C}$, the coefficient is $(m-2)$, and for $\mathfrak{sp}_m\mathbb{C}$, the coefficient is $(m+2)$.

Lecture 15

(15.19) See also Exercise 6.20.

(15.20) See Pieri's formulas (6.9), (6.8).

(15.21) Use the dimension formula (15.17).

(15.31) See Exercise 6.20.

(15.32) This is Exercise 6.16 in another notation (and restricted to the special linear group).

(15.33) See Exercise 6.16.

(15.51) Use Weyl's unitary trick with the group $\mathrm{U}(n)$.

(15.52) See Exercise 6.18.

(15.54) Show by induction on r that $r!$ times the difference is an integral linear combination of generators for $I^{\bullet}$. For details see [Tow2].

(15.57) The analogue of (15.53) is valid for these products of minors, and that can be used as in Proposition 15.55 to show that the e_T for semistandard T generate D_λ. The same e_T as in Proposition 15.55 is a highest weight vector. For more on this construction, see [vdW]; we learned it from J. Towber.

For other realizations of the representations of $\mathrm{GL}_n\mathbb{C}$, see [N-S].

Lecture 16

(16.7) With $v = (e_1 \wedge e_2)^2$, calculate as in §13.1; the two vectors $X_{2,1}V_2X_{2,1}V_2v$ and $X_{2,1}X_{2,1}V_2V_2v$ are proportional, and $V_2X_{2,1}X_{2,1}V_2v$ is independent of them.

Lecture 17

(17.18) (i) Note that $\Psi_{\{1,2\}}: \wedge^{s-2}V \to \wedge^s V$ is surjective if $s > n$. See Exercise 6.14 for the second statement. (ii) This can be done by direct calculation, as in [Wel, p. 155] for the harder case of the orthogonal group. Or, show that $\mathbb{S}_\lambda(V)$ has a highest weight vector with weight λ, and this cannot occur in any $\Psi_I(V^{\langle d-2\rangle})$.

(17.22) This follows from the theorem and the corresponding result for the general linear groups. Or see Exercise 6.30.

Lecture 19

(19.3)

$$V_{p,q}(v_I) = \begin{cases} 0 & \text{if } \{p, q\} \cap I = \varnothing \text{ or } \{p, q\} \subset I \\ \pm v_{I \setminus \{q\} \cup \{p\}} & \text{if } p \notin I \text{ and } q \in I \\ \pm v_{I \setminus \{p\} \cup \{q\}} & \text{if } q \notin I \text{ and } p \in I. \end{cases}$$

The first assertion follows readily. If $w = \sum a_I v_I$, with the fewest number of nonzero coefficients, and a_J and a_K are nonzero, choose $q \in J \setminus K$, $p \notin J \cup K$ (possible since $2k < m$); then $V_{p,q}(v_J) \neq 0$, $V_{p,q}(v_K) = 0$, and so $V_{p,q}(w)$ is a nonzero vector with fewer nonzero coefficients.

(19.4) The multiplicity of $L_1 + \cdots + L_a - L_{n-b} - \cdots - L_n$ in $\wedge^k V$ is $\binom{2r}{r}$ if $k - a - b = 2r$. For $\Gamma_{2\alpha}$ or $\Gamma_{2\beta}$ the multiplicity is $\frac{1}{2}\binom{2r}{r}$ if r is positive, by symmetry under replacing any Γ_p by $-\Gamma_p$. For Γ_α the weights are $\frac{1}{2}(\varepsilon_1 L_1 + \cdots + \varepsilon_n L_n)$, with $\varepsilon_i = \pm 1$, and $\prod \varepsilon_i = 1$; the multiplicities are all one since these are conjugate under the Weyl group; similarly for Γ_β but with $\prod \varepsilon_i = -1$.

(19.21) For generalizations, see §23.2.

Lecture 20

(20.17) If f spans $\wedge^n W'$, and u_0 spans U with $Q(u_0, u_0) = 1$, then $f \cdot (1 + (-1)^n u_0)$ is such a generator. See Exercise 20.12.

(20.21) If x is in the center, take an orthogonal basis $\{v_i\}$, write out $x = \sum a_I v_I$ in terms of the basis, and look at the equations $x \cdot v_j = v_j \cdot x$ for all j. Note that $v_I \cdot v_j = (-1)^{|I|} v_j \cdot v_I$ if $j \notin I$, whereas $v_I \cdot v_j = (-1)^{|I|-1} v_j \cdot v_I$ if $j \in I$. Conclude that $a_I = 0$ if $|I|$ is odd and there is some $j \notin I$ or if $|I|$ is even and there is some $j \in I$. A similar argument works if x is odd. Reference [A-B-S, p. 7].

(20.22) If $X = a \wedge b$, $[X, v] = \frac{1}{2}(a \cdot b \cdot v - b \cdot a \cdot v - v \cdot a \cdot b + v \cdot b \cdot a)$, which is

$$\tfrac{1}{2}(2Q(b, v)a - a \cdot v \cdot b - 2Q(a, v)b + b \cdot v \cdot a - 2Q(a, v)b + a \cdot v \cdot b + 2Q(b, v)a - b \cdot v \cdot a)$$

$$= 2Q(b, v)a - 2Q(a, v)b = \varphi_{a \wedge b}(v).$$

(20.23) Reference: [Por], but note that his $C(p, q)$ is our $C(q, p)$. See also [A-B-S].

(20.32) If $Q(v-w, v-w) \neq 0$, then $R_{v-w}(v) = w$. Otherwise, $R_{v+w}(v) = -w$, and $R_w(-w) = w$. For (b) compose a given element of $O(Q)$ with an element constructed by (a) to get one fixed on a line, and write, by induction on the dimension, the restriction to the perpendicular hyperplane as a product of reflections.

(20.33) By Exercise 20.22, $X \cdot v = [X, v]$. See also Exercise 8.24.

(20.36) If v_i are a basis for V with $Q(v_i, v_j) = -\delta_{i,j}$, then $\omega = v_1 \cdot \ldots \cdot v_m$. If $m \equiv 2$ (4), the center is cyclic of order four, while if $m \equiv 0$ (4), it is the Klein four group.

(20.37) Show that $\mathfrak{so}(Q)$ acts by traceless endomorphisms. For example, the trace of H_i on S^+ is the number of $I \subset \{1, \ldots, n\}$ such that $|I|$ is even and $i \in I$, minus the number with $i \notin I$.

(20.38) For the first statement of (a), choose f spanning $\wedge^n W'$ so that, for the chosen generator of $\wedge^n W$, $\tau(f) \cdot e \cdot f = f$. For the second, when m is even, $x(s)f = x \cdot s \cdot f$ by Exercise 20.12, so $\beta(x(s), x(t))f = \tau(x \cdot s \cdot f) \cdot (x \cdot t \cdot f) = \tau(s \cdot f) \cdot \tau(x) \cdot x \cdot (t \cdot f) = \tau(s \cdot f) \cdot (t \cdot f) = \beta(s, t)$. The odd case can be reduced to the even case by imbedding $C(Q)$ into a larger Clifford algebra as in Exercise 20.40.

(20.43) Reference: [Por].

(20.44) For example, the transposition of α_1 and α_4 is achieved by the matrix

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(20.50) Reference: [Ch2, §4.3].

(20.51) Reference: [Ch2, §4.2–4.5], [Jac1].

Other references include [L-M], [Ca1], [B-tD], [Hus], [P-S].

Lecture 21

(21.9) If $\alpha_1, \ldots, \alpha_r$ are the vectors, and we have a nontrivial relation

$$v = \sum_{i \leq k} n_i \alpha_i = \sum_{j > k} n_j \alpha_j,$$

with non-negative coefficients, then $(v, v) = \sum_{i,j} n_i n_j (\alpha_i, \alpha_j) \leq 0$, so $v = 0$. But v lies on the same side of the hyperplane.

(21.15) The first is ruled out by considering

$$u = e_2, \qquad v = (3e_3 + 2e_4 + e_5)/\sqrt{6}, \qquad w = (3e_6 + 2e_7 + e_8)/\sqrt{6},$$

with $1 > (e_1, u)^2 + (e_1, v)^2 + (e_1, w)^2 = 1/4 + 3/8 + 3/8 = 1$. For the second, use

$$u = e_2, \qquad v = (2e_3 + e_4)/\sqrt{3}, \qquad w = (5e_5 + 4e_6 + 3e_7 + 2e_8 + e_9)/\sqrt{15},$$

with $(e_1, u)^2 + (e_1, v)^2 + (e_1, w)^2 = 1/4 + 1/3 + 5/12 = 1$.

(21.16) Using the characterization that $\omega_i(H_{\alpha j}) = \delta_{i,j}$, one can write the fundamental weights ω_i in terms of the basis L_i. The tables in [Bour, Ch. 6] also express them in terms of the simple roots.

(E6):
$$\omega_1 = 2\frac{\sqrt{3}}{3}L_6,$$
$$\omega_2 = \tfrac{1}{2}(L_1 + L_2 + L_3 + L_4 + L_5) + \frac{\sqrt{3}}{2}L_6,$$
$$\omega_3 = \tfrac{1}{2}(-L_1 + L_2 + L_3 + L_4 + L_5) + 5\frac{\sqrt{3}}{6}L_6,$$
$$\omega_4 = L_3 + L_4 + L_5 + \sqrt{3}L_6,$$
$$\omega_5 = L_4 + L_5 + 2\frac{\sqrt{3}}{3}L_6,$$
$$\omega_6 = L_5 + \frac{\sqrt{3}}{3}L_6;$$

(E_7):
$$\omega_1 = \sqrt{2}L_7,$$
$$\omega_2 = \tfrac{1}{2}(L_1 + L_2 + L_3 + L_4 + L_5 + L_6) + \sqrt{2}L_7,$$
$$\omega_3 = \tfrac{1}{2}(-L_1 + L_2 + L_3 + L_4 + L_5 + L_6) + 3\frac{\sqrt{2}}{2}L_7,$$
$$\omega_4 = L_3 + L_4 + L_5 + L_6 + 2\sqrt{2}L_7,$$
$$\omega_5 = L_4 + L_5 + L_6 + 3\frac{\sqrt{2}}{2}L_7,$$
$$\omega_6 = L_5 + L_6 + \sqrt{2}L_7,$$
$$\omega_7 = L_6 + \frac{\sqrt{2}}{2}L_7;$$

(E_8)
$$\omega_1 = 2L_8,$$
$$\omega_2 = \tfrac{1}{2}(L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + 5L_8),$$
$$\omega_3 = \tfrac{1}{2}(-L_1 + L_2 + L_3 + L_4 + L_5 + L_6 + L_7 + 7L_8),$$
$$\omega_4 = L_3 + L_4 + L_5 + L_6 + L_7 + 5L_8,$$
$$\omega_5 = L_4 + L_5 + L_6 + L_7 + 4L_8,$$
$$\omega_6 = L_5 + L_6 + L_7 + 3L_8,$$
$$\omega_7 = L_6 + L_7 + 2L_8,$$
$$\omega_8 = L_7 + L_8;$$

(F_4):
$$\omega_1 = L_1 + L_2,$$
$$\omega_2 = 2L_1 + L_2 + L_3,$$
$$\omega_3 = \tfrac{1}{2}(3L_1 + L_2 + L_3 + L_4),$$
$$\omega_4 = L_1;$$

(G_2)
$$\omega_1 = \tfrac{1}{2}(L_1 + \sqrt{3}L_2) = 2\alpha_1 + \alpha_2,$$
$$\omega_2 = \sqrt{3}L_2 = 3\alpha_1 + 2\alpha_2.$$

(21.17) The only cases of the same rank that have the same number of roots are (B_n) and (C_n) for all n, and (B_6), (C_6), and (E_6); (B_n) has n roots shorter than the others, (C_n) n roots longer, and in (E_6) all the roots are the same length.

(21.18) For the matrices see [Bour, Ch. 6] or [Hu1, p. 59]. The determinants are:

$n + 1$ for (A_n); 2 for (B_n), (C_n) and (E_7); 4 for (D_n); 3 for (E_6); and 1 for (G_2), (F_4), and (E_8).

(21.23) See Lecture 22.

The proof of Lemma 21.20 is from [Jac1, p. 124], where details can be found.

For more on Dynkin diagrams and classification, see [Ch3], [Dem], [Dy-O], [LIE], and [Ti1].

Lecture 22

(22.5) Use the fact that $B(Y, Z) = 6\,\mathrm{Tr}(Y \circ Z)$ on $\mathfrak{sl}_3\mathbb{C}$, and the formula $[e_i, e_j^*] = 3 \cdot E_{i,j} - \delta_{i,j} \cdot I$, giving

$$B([e_i, e_j^*], Z) = 6 \cdot \mathrm{Tr}((3 \cdot E_{i,j} - \delta_{i,j} \cdot I) \circ Z) = 18 \cdot \mathrm{Tr}(E_{i,j} \circ Z) = 18 \cdot e_j^*(Z \cdot e_i).$$

(22.13) Hint: use the dihedral group symmetry.

(22.15) Answer: $\mathfrak{sl}_3\mathbb{C} \times \mathfrak{sl}_3\mathbb{C}$.

(22.20) For (b), apply ψ to both sides of (22.17), and evaluate both sides of (22.18) on w. Note that $\psi((v \wedge w) \wedge \varphi) = (v \wedge w)(\varphi \wedge \psi) = ((\varphi \wedge \psi) \wedge v)(w)$.

(22.21) For a triple $J = \{p < q < r\} \subset \{1, \ldots, 9\}$, let $e_J = e_p \wedge e_q \wedge e_r$ and similarly for φ_J. For triples J and K the essential calculation (see Exercise 22.5) is to verify that $e_J * \varphi_K$ is 1/18 times

$$\begin{array}{ll} 0 & \text{if } \# J \cap K \leq 1; \\ \pm E_{m,n} & \text{if } K = \{p, q, n\},\ J = \{p, q, m\},\ m \neq n; \\ E_{p,p} + E_{q,q} + E_{r,r} - \frac{1}{3}I & \text{if } K = J = \{p, q, r\}; \end{array}$$

the sign in front of $E_{m,n}$ is the product of the signs of the permutations that put the two sets in order. Verify that $(v \wedge w) \wedge \varphi = 18((w * \varphi) \cdot v - (v * \varphi) \cdot w)$. For Freudenthal's construction, see [Fr2], [H-S].

(22.24) For $\mathfrak{sl}_{n+1}\mathbb{C}$, such an involution takes $E_{i,j}$ to $(-1)^{j-i+1}E_{n+2-j,n+2-i}$; the fixed algebra is $\{X: {}^tXM = -MX\}$, where $M = (m_{ij})$, with $m_{ij} = 0$ if $i + j \neq n + 2$, and otherwise $m_{ij} = (-1)^i$. This M is symmetric if n is even, skew if n is odd, so the fixed subalgebra for (A_{2m}) is the Lie algebra $\mathfrak{so}_{2m+1}\mathbb{C}$ of (B_m), and that for (A_{2m-1}) is the Lie algebra $\mathfrak{sp}_{2m}\mathbb{C}$ of (C_m). For (D_n), the fixed algebra is $\mathfrak{so}_{2n-1}\mathbb{C}$, corresponding to (B_{n-1}), while for the rotation of (D_4), the fixed algebra is $\mathfrak{g}_2$. For a description of possible automorphisms of simple Lie algebras, see [Jac1, §IX].

(22.25) Answer: For $\mathfrak{sl}_{n+1}\mathbb{C}$, $X \mapsto -X^t$. For $\mathfrak{so}_{2n}\mathbb{C}$, $n \geq 5$, $X \mapsto PXP^{-1}$, where P is the automorphism of $\mathbb{C}^{2n}$ that interchanges e_n and e_{2n} and preserves the other basic vectors. For the other automorphisms of $\mathfrak{so}_8\mathbb{C}$, see Exercise 20.44.

(22.27) References: [Her], [Jac3, p. 777], [Pos], [Hu1, §19.3].

(22.38) Reference [Ch2, §4.5], [Jac4, p. 131], [Jac1], [Lo, p. 104].

Lecture 23

(23.3) The map takes $z = x + iy$ to (u, v) with $u = x/\|x\|$, $v = y$.

(23.10) Since $\rho(\exp(\sum a_j H_j)) = (e^{a_1}, \ldots, e^{a_n}, e^{-a_1}, \ldots)$, to be in the kernel we must have $a_j = 2\pi i \cdot n_j$, and then $\exp(\sum a_j H_j) = (-1)^{\sum n_j}$.

(23.11) Note that the surjectivity of the fundamental groups is equivalent to the connectedness of $\pi^{-1}(H)$ when $\pi: \tilde{G} \to G$ is the universal covering, which is equivalent to the Cartan subgroup of $\tilde{G}$ containing the center of $\tilde{G}$.

(23.17) Note that $\Gamma(G) = \pi_1(H)$ surjects onto $\pi_1(G)$, and there is an exact sequence

$$0 \to \pi_1(G) \to \mathrm{Center}(\tilde{G}) \to \mathrm{Center}(G) \to 0.$$

(23.19) When m is odd, the representations are the representations of $\mathrm{SO}_m\mathbb{C}$, and the products of those by the one-dimensional alternating (determinant) representation. When $m = 2n$, the representations of $\mathrm{SO}_m\mathbb{C}$ with highest weights $(\lambda_1, \ldots, \lambda_n)$ and $(\lambda_1, \ldots, -\lambda_n)$ are conjugate, so that, if $\lambda_n \neq 0$, they correspond to one irreducible representation of $\mathrm{O}_{2n}\mathbb{C}$, whose underlying space can be identified with $\Gamma_{(\lambda_1, \ldots, \lambda_n)} \oplus \Gamma_{(\lambda_1, \ldots, -\lambda_n)}$. If $\lambda_n = 0$, then Γ_λ is an irreducible representation of $\mathrm{O}_m\mathbb{C}$. In either case, the representations correspond to partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$. See §19.5 for an argument.

(23.31) See Exercises 19.6, 19.7, and 19.16.

(23.36) For (b), consider $(D^+)^2 \cdot (D^-)^2 = (D^+ \cdot D^-)^2$.

(23.37) Reference: [B-tD, VI §7].

(23.38) For $\mathfrak{sl}_{n+1}\mathbb{C}$, $\Gamma_\lambda^* = \Gamma_{(\lambda_1, \lambda_1 - \lambda_n, \ldots, \lambda_1 - \lambda_2)}$; for $\mathfrak{so}_{2n}\mathbb{C}$, n odd, $\Gamma_\lambda^* = \Gamma_{(\lambda_1, \lambda_2, \ldots, \lambda_{n-1}, -\lambda_n)}$.

(23.39) Reference: [Bour, VIII §7, Exer. 11].

(23.42) Compute highest weight vectors in the (external) tensor product of two irreducible representations, to verify that it is irreducible with highest weight the sum of the two weights.

(23.43) See Exercise 20.40 and Theorems 17.5 and 19.2.

(23.51) An isotropic $(n - 1)$-plane is automatically contained in an isotropic n-plane. These are two-step flag varieties, corresponding to omitting two nodes.

(23.62) For (b), use the fact that $B \cdot n' \cdot B$ is open in G. For (c), if μ is a weight, $f(x^{-1}wy) = \mu(x)\lambda(y)f(w)$ for x and y in B, so with $x \in H$ and $w = n'$,

$$\mu(x)f(w) = f(x^{-1}w) = f(wx) = \lambda(x)f(w).$$

Other references on homogeneous spaces include [B-G-G], [Hel], and [Hi].

Lecture 24

(24.4) (a) is proved in Lemma D.25, and (b) follows. For (c), note that by the definition of ρ as half the sum of the positive roots, $\rho - W(\rho)$ is the sum of those positive β such that $W(\beta)$ is negative.

(24.27) This is Exercise A.62.

(24.46) This follows from formulas (A.61) and (A.65).

(24.51) In the following the fundamental weights are numbered as in the answer to Exercise 21.16:

(F_4):
$$\rho = 8\alpha_1 + 15\alpha_2 + 21\alpha_3 + 11\alpha_4;$$
$$\dim(\Gamma_{\omega_i}),\ (i = 1, 2, 3, 4)\text{:}\quad 52,\ 1274,\ 273,\ 26.$$

(E_6):
$$\rho = L_2 + 2L_3 + 3L_4 + 4L_5 + 4\sqrt{3}L_6$$
$$= 8\alpha_1 + 11\alpha_2 + 15\alpha_3 + 21\alpha_4 + 15\alpha_5 + 8\alpha_6;$$
$$\dim(\Gamma_{\omega_i}),\ (i = 1, \ldots, 6)\text{:}\quad 27,\ 78,\ 351,\ 2925,\ 351,\ 27.$$

(E_7):
$$\rho = L_2 + 2L_3 + 3L_4 + 4L_5 + 5L_6 + 17\sqrt{2}/2L_7$$
$$= \tfrac{1}{2}(34\alpha_1 + 49\alpha_2 + 66\alpha_3 + 96\alpha_4 + 75\alpha_5 + 52\alpha_6 + 27\alpha_7);$$
$$\dim(\Gamma_{\omega_i}),\ (i = 1, \ldots, 7)\text{:}\quad 133,\ 912,\ 8645,\ 365750,\ 27664,\ 1539,\ 56.$$

(E_8):
$$\rho = L_2 + 2L_3 + 3L_4 + 4L_5 + 5L_6 + 6L_7 + 23L_8$$
$$= 46\alpha_1 + 68\alpha_2 + 91\alpha_3 + 135\alpha_4 + 110\alpha_5 + 84\alpha_6 + 57\alpha_7 + 29\alpha_8;$$
$$\dim(\Gamma_{\omega_i}),\ (i = 1, \ldots, 8)\text{:}\quad 3875,\ 147250,\ 6696000,\ 6899079264,\ 146325270,$$
$$2450240,\ 30380,\ 248.$$

(24.52) Using the dimension formula as in Exercise 24.9, it suffices to check which fundamental weights correspond to small representations, and then which sums of these are still small. The results are:

(A) $n \geq 1$; $\dim G = n^2 + 2n$; $\dim \Gamma_{\omega_k} = \binom{n+1}{k}$;

the dominant weights whose representations have dimension at most $\dim G$ are:

$\omega_1, \omega_2, \omega_{n-1}, \omega_n$;

$2\omega_1, 2\omega_n$, of dimension $\binom{n+2}{2}$;

$\omega_1 + \omega_n$, of dimension $n^2 + 2n$;

ω_3 for $n = 5$; ω_3, ω_4 for $n = 6$; ω_3, ω_5 for $n = 7$.

(B_n) $n \geq 2$; $\dim G = 2n^2 + n$; $\dim \Gamma_{\omega_k} = \binom{2n+1}{k}$ for $k < n$, and $\dim \Gamma_{\omega_n} = 2^n$, giving:

ω_1, ω_2;

ω_n for $n = 3, 4, 5, 6$;

$2\omega_2$, of dimension 10, for $n = 2$.

(C_n) $n \geq 3$; $\dim G = 2n^2 + n$; $\dim \Gamma_{\omega_k} = \binom{2n}{k} - \binom{2n}{k-2}$, giving:

ω_1, ω_2;

$2\omega_1$, of dimension $2n^2 + n$;

ω_3 for $n = 3$.

(D_n) $n \geq 4$; $\dim G = 2n^2 - n$; $\dim \Gamma_{\omega_k} = \binom{2n}{k}$ for $k \leq n-2$, and

$\dim \Gamma_{\omega_{n-1}} = \dim \Gamma_{\omega_n} = 2^{n-1}$, giving:

ω_1, ω_2;

ω_{n-1}, ω_n for $n = 4, 5, 6, 7$.

(E_6) $\dim G = 78$; $\omega_1, \omega_2, \omega_6$.

(E_7) $\dim G = 133$; ω_1, ω_7.

(E_8) $\dim G = 248$; ω_8.

(F_4) $\dim G = 52$; ω_1, ω_4.

(G_2) $\dim G = 14$; ω_1, ω_2.

For irreducible representations of general Lie groups with this property, see [S-K].

Other references with character formulas include [ES-K], [Ki1], [Ki2], [Kl], [Mur2], and [Ra].

Lecture 25

(25.2) Changing μ by an element of the Weyl group, one can assume μ is also dominant and $\lambda - \mu$ is a sum of positive roots. Then $\|\lambda\| > \|\mu\|$, and $c(\mu) = (\lambda, \lambda) - (\mu, \mu) + (\lambda - \mu, 2\rho) > 0$.

(25.4) A direct calculation gives

$$C(X \cdot v) - X \cdot C(v) = \sum U_i \cdot [U_i', X] \cdot v + \sum [U_i, X] \cdot U_i' \cdot v.$$

To see that this is zero, write $[U_i, X] = \sum \alpha_{ij} U_j$; then by (14.23), $\alpha_{ij} = ([U_i, X], U_j') = -([U_j', X], U_i)$, so $[U_j', X] = -\sum \alpha_{ij} U_i'$. The terms in the above sums then cancel in pairs.

(25.6) By (14.25), $(H_\alpha, H_\alpha) = \alpha(H_\alpha)(X_\alpha, Y_\alpha) = 2(X_\alpha, Y_\alpha)$. Use Exercise 14.28.

(25.12) The symmetry gives

$$(\beta - i\alpha, \alpha) n_{\beta - i\alpha} + (\beta - (m-i)\alpha, \alpha) n_{\beta - (m-i)\alpha} = (2\beta - m\alpha, \alpha) n_{\beta - i\alpha} = 0$$

since $2(\beta, \alpha) = m(\alpha, \alpha)$, so the terms cancel in pairs.

(25.22) We have

$$\sum_{W, \mu} (-1)^W P(\mu + W(\rho) - \rho) e(-\mu) = \sum_{W} (-1)^W (e(W(\rho) - \rho)) / \prod_{\alpha \in R^+} (1 - e(-\alpha)),$$

and the right-hand side is 1 by Lemma 24.3.

(25.23) We have

$$\begin{aligned} \sum_W (-1)^W n_{\mu + \rho - W(\rho)} &= \sum_{W, W'} (-1)^{WW'} P(W'(\lambda + \rho) - ((\mu + \rho - W(\rho)) + \rho)) \\ &= \sum_{W'} (-1)^{W'} \sum_{W} (-1)^W P((W'(\lambda + \rho) - \mu - \rho) + W(\rho) - \rho), \end{aligned}$$

and the inner sum is zero unless $W'(\lambda + \rho) = \mu + \rho$. Note that if μ is a root of Γ_λ, this happens only if $\mu = \lambda$ by Exercise 25.2.

(25.24) The minuscule weights are:

(A_n):	$\omega_1, \ldots, \omega_n$,
(B_n):	ω_1,
(C_n):	ω_n,
(D_n):	$\omega_1, \omega_{n-1}, \omega_n$,
(E_6):	ω_1, ω_6,
(E_7):	ω_7.

Reference: [Bour, VIII, §7.3].

(25.28) One easy way is to use the isomorphism $\mathfrak{so}_4\mathbb{C} \cong \mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$.

(25.30) $N_{\lambda\mu\gamma}$ is zero by definition when γ is not in the closed positive Weyl chamber $\mathscr{W}$, and $W(\nu + \rho) - \rho$ is not in $\mathscr{W}$ if $W \neq 1$. Reference: [Hu1].

(25.40) The weight space of the restriction of Γ_λ corresponding to $\bar{\mu}$ is the direct sum of the weight spaces of Γ_λ corresponding to those μ which restrict to $\bar{\mu}$.

(25.41) Use the preceding exercise and Exercise 25.23.

(25.43) Using the action of a Lie algebra on a tensor product, the action of C on $v_1 \cdot \ldots \cdot v_m$ is a sum over terms where U_i and U_i' act on different elements or the same element. Grouping the terms accordingly leads to the displayed formula. See [L-T, I, pp. 19–20].

Lecture 26

(26.2) In terms of the basis L_1, L_2 of $\mathfrak{h}^*$ dual to $\{H_1, H_2\}$, eigenvalues are $\pm iL_2$ and $\pm 3L_1 \pm iL_2$.

(26.9) Reference: [Hel, §III.7].

(26.10) Constructing $\mathfrak{h} = \mathfrak{g}_0(H)$ as in Appendix D, take H so that $\sigma(H) = H$.

(26.12) See Exercise 23.6.

(26.13) Reference: [Hel, §X.6.4].

(26.21) If a conjugate linear endomorphism φ: $W \to W$ did not map Γ_λ to itself, there would be another factor U of W and an isomorphism of Γ_λ with U^*; the highest weight of $(\Gamma_\lambda)^*$ cannot be lower than λ.

(26.22) See Exercise 3.43 and Exercise 26.21.

(26.28) References: [A-B-S], [Hus], [Por]. See also Exercise 20.38.

(26.30) Use the identity $\psi^2[V] = [V \otimes V] - 2[\wedge^2 V]$.

Other references on real forms are [Gi1], [B-tD], [Va].

Appendix A

(A.29) (b) Use $P^{(i)} = \sum_{\nu} \langle H_{\nu}, P^{(i)} \rangle M_{\nu}$.

(A.30) Some of these formulas also follow from Weyl's character formula.

(A.31) For part (a), when $a_1 \geq a_2 \geq \cdots \geq a_k$, this is (A.19). The proof of (A.9) shows that for any $a = (a_1, \ldots, a_k)$,

$$H_{a_1} \cdot H_{a_2} \cdot \ldots \cdot H_{a_k} = \sum K_{\mu a} S_{\mu},$$

which shows that the $K_{\mu a}$ are unchanged when the a_i's are reordered. For a purely combinatorial proof see [Sta, §10].

(A.32) For (i) compare the generating functions $E(t) = \sum E_i t^i = \prod (1 + x_i t)$ and $H(t) = \sum H_i t^i = 1/E(-t)$; (ii) follows from (A.5) and (A.6). For (iii), note that $P(t) = \sum P_j t^j = \sum x_i t/(1 - x_i t) = tH'(t)/H(t)$. Exponentiate this to get (vi). For details and more on this involution, see [Mac] or [Sta], where it is used to derive basic identities among symmetric polynomials.

(A.39) References: [Mac], [Sta], [Fu, §A.9.4].

(A.41) See [Mac, p. 33] or [Fu, p. 420].

(A.48) Since $\vartheta(E_i') = H_i'$ and $\vartheta(E_i'') = H_i''$,

$$\vartheta(S_{\langle \lambda \rangle}) = \vartheta(|H''_{\lambda_i - i + j} - H''_{\lambda_i - i - j}|) = |E''_{\mu_i - i + j} - E''_{\mu_i - i - j}| = S_{[\mu]}.$$

(A.67) Answer: $\frac{1}{2}\zeta_1 \cdot \ldots \cdot \zeta_n$ times the determinant of the matrix whose ith row is

$$(J_{\lambda_i - i} \quad J_{\lambda_i - i + 1} + J_{\lambda_i - i - 1} \quad \cdots \quad J_{\lambda_i - i + n - 1} + J_{\lambda_i - i - n + 1}).$$

More on symmetric polynomials can be found in [Mac], [Sta], [L-S], and references listed in these sources. Some of the identities in §A.3 are new, although results along these lines can be found in [We1], [Lit1], [Lit2] and [Ko-Te]; other identities involving the determinants discussed in §A.3 can be found in [Mac, §I.5]. Discussions of Schur functions and representation theory can be found in [Di2] and [Lit2].

Appendix C

(C.1) Take a basis in which X has Jordan canonical form, and compute using the corresponding basis E_{ij} for $\mathfrak{gl}(V)$.

(C.12) If $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, and $\mathfrak{h}$ is a simple ideal, $\mathfrak{h} = [\mathfrak{g}, \mathfrak{h}] = \bigoplus [\mathfrak{g}_i, \mathfrak{h}]$, so $\mathfrak{h}$ is contained in some $\mathfrak{g}_i$.

(C.13) Since for $\delta \in \mathrm{Der}(\mathfrak{g})$ and $X \in \mathfrak{g}$, $\mathrm{ad}(\delta(X)) = [\delta, \mathrm{ad}(X)]$, $\mathrm{ad}(\mathfrak{g})$ is an ideal in the Lie algebra $\mathrm{Der}(\mathfrak{g})$. Therefore, $[\mathrm{ad}(\mathfrak{g})^{\perp}, \mathrm{ad}(\mathfrak{g})] = 0$; in particular, if $\delta \in \mathrm{ad}(\mathfrak{g})^{\perp}$ and $X \in \mathfrak{g}$, then $\mathrm{ad}(\delta(X)) = [\delta, \mathrm{ad}(X)] = 0$. So $\mathrm{ad}(\mathfrak{g})^{\perp} = 0$ and $\mathrm{ad}(\mathfrak{g}) = \mathrm{Der}(\mathfrak{g})$.

Appendix D

(D.8) To show $\mathrm{ad}(X)$ is nilpotent on $\mathfrak{g}_0(H)$ for X in $\mathfrak{g}_0(H)$, consider the complex line from H to X: set $H(z) = (1 - z)H + zX$. Then $\mathrm{ad}(H(z))$ preserves each eigenspace $\mathfrak{g}_{\lambda}(H)$. By continuity, for z sufficiently near 0, $\mathrm{ad}(H(z))$ is a nonsingular transformation of $\mathfrak{g}_\lambda(H)$ for $\lambda \neq 0$, which implies that $\mathfrak{g}_0(H(z))$ is contained in $\mathfrak{g}_0(H)$, and by the regularity of H, $\mathfrak{g}_0(H(z)) = \mathfrak{g}_0(H)$ for small z.

This means that there is an integer k so that $\mathrm{ad}(H(z))^k(Y) = 0$ for all $Y \in \mathfrak{g}_0(H)$ and all small z. But $\mathrm{ad}(H(z))^k(Y)$ is a polynomial function of z, so it must vanish identically. Hence, setting $z = 1$, $\mathrm{ad}(X)^k$ vanishes on $\mathfrak{g}_0(H)$, as asserted.

(D.24) See [Bour, VII, §3] for details.

(D.33) References: [Se3, §V.11], [Hu1, §12.2].

Appendix E

Proofs of both of these theorems can be found, together with many other related results, in [Bour I]. See also [Se3], [Pos], [Va], [Jac1].

Appendix F

(F.12) Check that the right-hand side is multilinear, alternating, and takes the value 1 on a standard basis. Or see [We1, §VI.1].

(F.16) $\mathrm{SO}_n\mathbb{C}$-invariants can be written in the form $A + \sum A_i B_i$ where A and the A_i are polynomials in the $Q(x^{(i)}, x^{(j)})$ and the B_i are brackets. Such is taken to $A + \det(g)\sum A_i B_i$ by g in $\mathrm{O}_n\mathbb{C}$. For an odd (resp. even) invariant the first (resp. the second) term must vanish.

(F.20) Reference: [We1, II.6], or [Br, p. 866].

There are many elementary references for invariant theory, such as [D-C], [Pr], [Sp1], and [Ho2]; the last contains a proof of Capelli's formula. There are also many modern approaches to invariant theory, some which can be found in [DC-P], [Sch] and [Vu] and references described therein; some of these also contain some invariant theory for exceptional groups. For a more conceptual and representation-theoretic approach to Capelli's identity, see [Ho3]. Weyl's book [We1] remains an excellent reference for invariant theory of the orthogonal and symplectic groups together with the related [Br], [We2].

Bibliography

[A-B] M. F. Atiyah and R. Bott, A Lefschetz fixed point formula for elliptic complexes: II. Applications, *Ann. Math.* **9** (1968), 451–491.

[A-B-S] M. F. Atiyah, R. Bott, and A. Shapiro, Clifford modules, *Topology* **3**, Supp. 1 (1964), 3–38.

[A-B-W] K. Akin, D. A. Buchsbaum, and J. Weyman, Schur functors and Schur complexes, *Adv. Math.* **44** (1982), 207–278.

[Ad] J. F. Adams, *Lectures on Lie Groups*, W. A. Benjamin, Inc., New York, 1969.

[Ahl] L. V. Ahlfors, *Complex Analysis, Second Edition*, McGraw-Hill, New York 1966.

[A-J-K] Y. J. Abramsky, H. A. Jahn, and R. C. King, Frobenius symbols and the groups S_s, GL(n), O(n), and Sp(n), *Can. J. Math.* **25** (1973), 941–959.

[And] G. E. Andrews, The Theory of Partitions, *Encyclopedia of Mathematics and Its Applications*, vol. 2, Addison-Wesley, Reading, MA, 1976.

[Ar] S. K. Araki, On root systems and an infinitesimal classification of irreducible symmetric spaces, *J. Math. Osaka City Univ.* **13** (1963), 1–34.

[A-T] M. Atiyah and D. O. Tall, Group representations, λ-rings, and the J-homomorphism, *Topology* **8** (1969), 253–297.

[B-G-G] I. N. Bernstein, I. M. Gelfand, and S. I. Gelfand, Schubert cells and cohomology of the spaces G/P, *Russ. Math. Surv.* **28** (1973), 1–26.

[Boe] H. Boerner, *Representations of Groups*, Elsevier North-Holland, Amsterdam, 1970.

[Bor1] A. Borel, *Linear Algebraic Groups*, W. A. Benjamin, 1969 and (GTM 126), Springer-Verlag, New York, 1991.

[Bor2] A. Borel, Topology of Lie groups and characteristic classes, *Bull. Amer. Math. Soc.* **61** (1955), 397–432.

[Bot] R. Bott, On induced representations, in *The Mathematical Heritage of Hermann Weyl*, Proc. Symp. Pure Math Vol. 48, American Mathematical Society, Providence, RI 1988, pp. 1–13.

[Bour] N. Bourbaki, *Lie Groups and Lie Algebras, Chapters 1–3*, Springer-Verlag, New York, 1989; *Groupes et algèbres de Lie, Chapitres 4, 5 et 6*, Masson, Paris, 1981; *Groupes et algèbres de Lie, Chapitres 7 et 8*, Diffusion C.C.L.S., Paris, 1975; *Algebra 1*, Chapter 3, Springer-Verlag, New York, 1989.

[Br] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Ann. Math.* **38** (1937), 857–872.

[B-tD] T. Bröcker and T. tom Dieck, *Representations of Compact Lie Groups*, Springer-Verlag, New York, 1985.

[Bu] J. Burroughs, Operations in Grothendieck rings and the symmetric group, *Can. J. Math.* **26** (1974), 543–550.

[Ca1] E. Cartan, *The Theory of Spinors*, Hermann, Paris, 1966, and Dover Publications, 1981.

[Ca2] E. Cartan, Le principe de dualité et la théorie des groupes simples et semi-simples, Bull. Sci. Math. **49** (1925), 361–374.

[Cart] P. Cartier, On H. Weyl's character formula, *Bull. Amer. Math. Soc.* **67** (1961), 228–230.

[Ch1] C. Chevalley, *Theory of Lie Groups*, Princeton University Press, Princeton, NJ, 1946.

[Ch2] C. Chevalley, *The Algebraic Theory of Spinors*, Columbia University Press, New York, 1954.

[Ch3] Séminaire C. Chevalley 1956–1958, *Classification des Groupes de Lie Algébriques, Secrétariat mathématique*, Paris, 1958.

[Ch-S] C. Chevalley and R. D. Schafer, The exceptional simple Lie algebras F_4 and E_6, *Proc. Natl. Acad. Sci. USA.* **36** (1950), 137–141.

[Co] A. J. Coleman, *Induced Representations with Applications to S_n and* GL(n), *Queens Papers Pure Appl. Math.* **4** (1966).

[C-R] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience Publishers, New York, 1962.

[D-C] J. Dieudonné and J. Carrell, *Invariant Theory, Old and New*, Academic Press, New York, 1971.

[DC-P] C. De Concini and C. Procesi, A characteristic free approach to invariant theory, *Adv. Math.* **21** (1976), 330–354.

[Dem] M. Demazure, *A, B, C, D, E, F*, etc., *Springer Lecture Notes* 777, Springer-Verlag, Heidelberg, 1980, pp. 221–227.

[D'H] E. D'Hoker, Decompositions of representations into basis representations for the classical groups, J. Math. Physics **25** (1984), 1–12.

[Di1] J. Dieudonné, *Sur les Groupes Classiques*, Hermann, Paris, 1967.

[Di2] J. Dieudonné, Schur functions and group representations, in *Young tableaux and Schur functors in algebra and geometry, Astérisque* **87–88** (1981), 7–19.

[Dia] P. Diaconis, *Group Representations in Probability and Statistics*, Institute of Mathematical Statistics, Hayward, CA, 1988.

[Don] R. Donagi, On the geometry of Grassmannians, *Duke Math. J.* **44** (1977), 795–837.

[Dor] L. Dornhoff, *Group Representation Theory*, Parts *A* and *B*, Marcel Dekker, New York, 1971, 1972.

[Dr] D. Drucker, Exceptional Lie algebras and the structure of hermitian symmetric spaces, *Mem. Amer. Math. Soc.* **208** (1978).

[Dy-O] E. B. Dynkin and A. L. Oniščik, Compact global Lie groups, *Amer. Math. Soc. Transl., Series 2* **21** (1962), 119–192.

[ES-K] N. El Samra and R. C. King, Reduced determinantal forms for characters of the classical Lie groups, *J. Phys. A: Math. Gen.* **12** (1979), 2305–2315.

[Foa] D. Foata (ed.), *Combinatoire et Représentation du Groupe Symétrique, Strasbourg 1976*, Springer Lecture Notes 579, Springer-Verlag, Heidelberg, 1977.

[Fr1] H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Mathematisch Instituut der Rijksuniversiteit te Utrecht, Utrecht, 1951, 1960.

[Fr2] H. Freudenthal, Lie groups in the foundations of geometry, *Adv. Math.* **1** (1964), 145–190.

[Fr-dV] H. Freudenthal and H. de Vries, *Linear Lie Groups*, Academic Press, New York, 1969.

[Fro1] F. G. Frobenius, Über die Charaktere der symmetrischen Gruppe, *Sitz. König. Preuss. Akad. Wissen.* (1900), 516–534; Gesammelte Abhandlungen III, Springer-Verlag, Heidelberg, 1968, pp. 148–166.

[Fro2] F. G. Frobenius, Über die Charaktere der alternirenden Gruppe, *Sitz. König. Preuss. Akad. Wissen.* (1901), 303–315; Gesammelte Abhandlungen III, Springer-Verlag, Heidelberg, 1968, pp. 167–179.

[Fu] W. Fulton, *Intersection Theory*, Springer-Verlag, New York, 1984.

[G-H] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley-Interscience, New York, 1978.

[Gil] R. Gilmore, *Lie Groups, Lie Algebras, and Some of Their Applications*, Wiley, New York, 1974.

[G-N-W] C. Greene, A. Nijenhuis, and H. S. Wilf, A probabilistic proof of a formula for the number of Young tableaux of a given shape, *Adv. Math.* **31** (1979), 104–109.

[Gr] J. A. Green, The characters of the finite general linear group, *Trans. Amer. Math. Soc.* **80** (1955), 402–447.

[Gre] M. L. Green, Koszul cohomology and the geometry of projective varieties, I, II, *J. Diff. Geom.* **19** (1984), 125–171; **20** (1984), 279–289.

[Grie] R. L. Griess, Automorphisms of extra special groups and nonvanishing degree 2 cohomology, *Pacific J. Math.* **48** (1973), 403–422.

[Ha] J. Harris, *Algebraic Geometry*, Springer-Verlag, New York, to appear.

[Ham] M. Hamermesh, *Group Theory and its Application to Physical Problems*, Addison-Wesley, Reading, MA, 1962 and Dover, 1989.

[Har] G. H. Hardy, *Ramanujan*, Cambridge University Press, Cambridge, MA, 1940.

[Hel] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[Her] R. Hermann, *Spinors, Clifford and Cayley Algebras*, Interdisciplinary Mathematics Volume VII, 1974.

[Hi] H. Hiller, *Geometry of Coxeter Groups*, Pitman, London, 1982.

[Ho1] R. Howe, Very basic Lie theory, *Amer. Math. Monthly* **90** (1983), 600–623; **91** (1984), 247.

[Ho2] R. Howe, The classical groups and invariants of binary forms, *Proc. Symp. Pure Math.* **48** (1988), 133–166.

[Ho3] R. Howe, Remarks on classical invariant theory, *Trans. Amer. Math. Soc.* **313** (1989), 539–569.

[H-P] W. V. D. Hodge and D. Pedoe, *Methods of Algebraic Geometry*, vols. 1 and 2, Cambridge University Press, Cambridge, MA, 1947, 1952; 1968.

[H-S] M. Hausner and J. T. Schwarz, *Lie groups*; *Lie algebras*, Gordon and Breach, New York, 1968.

[Hu1] J. E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York, 1972, 1980.

[Hu2] J. E. Humphreys, *Linear Algebraic Groups*, Springer-Verlag, New York, 1975, 1981.

[Hu3] J. E. Humphreys, Representations of SL(2, p), *Amer. Math. Monthly* **82** (1975), 21–39.

[Hur] A. Hurwitz, Über die Anzahl der Riemann'schen Flächen mit gegebenen Verzweigungspunkten, *Math. Ann.* **55** (1902), 53–66.

[Hus] D. Husemoller, *Fibre Bundles*, second edition, Springer-Verlag, New York, 1975.

[In] R. E. Ingram, Some characters of the symmetric group, *Proc. Amer. Math. Soc.* **1** (1950), 358–369.

[Iv] B. Iversen, The geometry of algebraic groups, *Adv. Math.* **20** (1976), 57–85.

[Jac1] N. Jacobson, *Lie Algebras*, Wiley, New York 1962, and Dover, 1979.

[Jac2] N. Jacobson, *Exceptional Lie Algebras*, Marcel Dekker, New York 1971.

[Jac3] N. Jacobson, Cayley numbers and simple Lie algebras of type G, *Duke Math. J.* **5** (1939), 775–783.

[Jac4] N. Jacobson, Triality and Lie algebras of type D_4, *Rend. Circ. Mat. Palermo* (2) **13** (1964), 129–153.

[Ja-Ke] G. James and A. Kerber, *The Representation Theory of the Symmetric Group*, Encyclopedia of Mathematics and Its Applications, vol. 16, Addison-Wesley, Reading, MA, 1981.

[Jam] G. D. James, *The Representation Theory of the Symmetric Groups*, Springer Lecture Notes 682, Springer-Verlag, Heidelberg, 1978.

[J-L] T. Józefiak and A. Lascoux (eds.), *Young Tableaux and Schur Functors in Algebra and Geometry, Toruń, Poland* 1980, *Astérisque* 87–88, 1981.

[Ke] A. Kerber, *Representations of Symmetric Groups I*, Springer Lecture Notes 240, Springer-Verlag, Heidelberg, 1971.

[Kem] G. Kempf, Tensor products of representations, *Amer. J. Math.* **109** (1987), 401–415.

[Ki1] R. C. King, The dimensions of irreducible tensor representations of the orthogonal and symplectic groups, *Can. J. Math.* **23** (1971), 176–188.

[Ki2] R. C. King, Modification rules and products of irreducible representations of the unitary, orthogonal and symplectic groups, *J. Math. Phys.* **12** (1971), 1588–1598.

[Kir] A. A. Kirillov, *Elements of the Theory of Representations*, Springer-Verlag, New York, 1976.

[Kl] A. U. Klymyk, Multiplicities of weights of representations and multiplicities of representations of semisimple Lie algebras, *Sov. Math. Dokl.* **8** (1967), 1531–1534.

[K-N] G. Kempf and L. Ness, Tensor products of fundamental representations, *Can. J. Math.* **40** (1988), 633–648.

[Kn] D. Knutson, *λ-Rings and the Representation Theory of the Symmetric Group*, Springer Lecture Notes 308, Springer-Verlag, Heidelberg 1973.

[Kos] B. Kostant, A formula for the multiplicity of a weight, *Trans. Amer. Math. Soc.* **93** (1959), 53–73.

[Ko-Te] K. Koike and I. Terada, Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, and D_n, *J. Algebra* **107** (1987), 466–511.

[Ku] J. P. S. Kung (ed.), *Young Tableaux in Combinatorics, Invariant Theory, and Algebra*, Academic Press, New York, 1982.

[Kum1] S. Kumar, Proof of the Parthasarathy–Ranga Rao–Varadarajan conjecture, *Invent. Math.* **93** (1988), 117–130.

[Kum2] S. Kumar, A refinement of the PRV conjecture, *Invent. Math.* **97** (1989), 305–311.

[Le] W. Ledermann, *Introduction to Group Characters*, Cambridge University Press, Cambridge, MA, 1977.

[Li] P. Littelmann, A Littlewood–Richardson rule for classical groups, *C. R. Acad. Sci. Paris* **306** (1988), 299–303.

[LIE] Séminaire Sophus LIE 1954/1955, *Théorie des Algèbres de Lie, Topologie des groupes de Lie*, École Normale Supérieure, Paris, 1955.

[Lit1] D. E. Littlewood, *The Theory of Group Characters and Matrix Representations of Groups*, second ed., Oxford University Press, Oxford, 1950.

[Lit2] D. E. Littlewood, *A University Algebra*, William Heinemann Ltd, London, 1950; second ed., 1958, and Dover, 1970.

[Lit3] D. E. Littlewood, On invariants under restricted groups, *Philos. Trans. Roy. Soc. A* **239** (1944), 387–417.

[Liu] A. Liulevicius, Arrows, symmetries and representation rings, *J. Pure Appl. Algebra* **19** (1980), 259–273.

[L-M] H. B. Lawson and M.-L. Michelson, *Spin Geometry*, Princeton University Press, Princeton, NJ, 1989.

[L-M-S] V. Lakshmibai, C. Musili, and C. S. Seshadri, Geometry of G/P, *Bull. Amer. Math. Soc.* **1** (1979), 432–435.

[Lo] Loos, *Symmetric Spaces*, W. A. Benjamin, New York, 1969.

[L-S] A. Lascoux and M. P. Schützenberger, *Formulaire raisonné de fonctions symmetriques*, U. E. Maths, Paris VII, L.A. 248, 1985.

[L-T] G. Lancaster and J. Towber, Representation-functors and flag-algebras for the classical groups I, II, *J. Algebra* **59** (1979), 16–38; **94** (1985), 265–316.

[L-VdV] R. Lazarsfeld and A. Van de Ven, *Topics in the Geometry of Projective Space*, DMV Seminar Band 4, Birkhäuser, Boston, MA, 1984.

[L-V] R. A. Liebler and M. R. Vitale, Ordering the partition characters of the symmetric group, *J. Algebra* **25** (1973), 487–489.

[Mac] I. G. Macdonald, *Symmetric Functions and Hall Polynomials*, Clarendon Press, Oxford, 1979.

[Mack] G. W. Mackey, *Introduction to The Racah–Wigner Algebra in Quantum Theory*, by L. C. Biedenharn and J. D. Louck, Encyclopedia of Mathematics and Its Applications, vol. 9, Addison-Wesley, Reading, MA, 1981.

[M-S] G. Musili and C. S. Seshadri, *Standard monomial theory*, Springer Lecture Notes **867** (1981), 441–476.

[Mur1] F. D. Murnaghan, *The Theory of Group Representations*, The Johns Hopkins Press, Baltimore, 1938.

[Mur2] F. D. Murnaghan, *The Unitary and Rotation Groups*, Spartan Books, Washington, DC, 1962.

[No] K. Nomizu, *Lie Groups and Differential Geometry*, Mathematics Society of Japan, Tokyo, 1956.

[N-S] M. A. Naimark and A. I. Stern, *Theory of Group Representations*, Springer-Verlag, New York, 1982.

[Pe1] M. H. Peel, Hook representations of symmetric groups, *Glasgow Math. J.* **12** (1971), 136–149.

[Pe2] M. H. Peel, Specht modules and the symmetric groups, *J. Algebra* **36** (1975), 88–97.

[Por] I. R. Porteous, *Topological Geometry*, second edition, Cambridge University Press, Cambridge, MA, 1981.

[Pos] M. Postnikov, *Lie Groups and Lie Algebras*, MIR, Moscow 1986.

[Pr] C. Procesi, *A Primer of Invariant Theory*, Brandeis Lecture Notes 1, 1982.

[P-S] A. Pressley and G. Segal, *Loop Groups*, Clarendon Press, Oxford, 1986.

[P-W] P. Pragacz and J. Weyman, On the construction of resolutions of determinantal ideals: a survey, Springer Lecture Notes 1220 (1986), 73–92.

[Qu] D. Quillen, The mod 2 cohomology rings of extra-special 2-groups and the spinor groups, *Math. Ann.* **194** (1971), 197–212.

[Ra] G. Racah, Lectures on Lie groups, in *Group Theoretical Concepts and Methods in Elementary Particle Physics*, Gordon and Breach, New York, 1964, 1–36.

[Sc] R. D. Schafer, *An Introduction to Nonassociative Algebras*, Academic Press, New York, 1966.

[Sch] G. W. Schwarz, On classical invariant theory and binary cubics, *Ann. Inst. Fourier* **37** (1987), 191–216.

[Se1] J-P. Serre, *Lie Algebras and Lie Groups*, W. A. Benjamin, New York, 1965.

[Se2] J-P. Serre, *Linear Representations of Finite Groups*, Springer-Verlag, New York, 1977.

[Se3] J-P. Serre, *Complex Semi-simple Lie Algebras*, Springer-Verlag, New York, 1987.

[S-K] M. Sato and T. Kimura, A classification of irreducible prehomogeneous vector spaces and their relative invariants, *Nagoya Math. J.* **65** (1977), 1–155.

[Sp] T. A. Springer, *Invariant Theory*, Springer Lecture Notes 565, Springer-Verlag, Heidelberg, 1977.

[Sta] R. P. Stanley, Theory and Application of Plane Partitions, Parts 1 and 2, *Studies Appl. Math.* **1** (1971), 167–188, 259–279.

[Ste1] R. Steinberg, The representations of GL(3, q), GL(4, q), PGL(3, q), and PGL(4, q), *Can. J. Math.* **3** (1951), 225–235.

[Ste2] R. Steinberg, *Conjugacy classes in Algebraic Groups*, Springer Lecture Notes 366, Springer-Verlag, Heidelberg, 1974.

[S-W] D. H. Sattinger and O. L. Weaver, *Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics*, Springer-Verlag, New York, 1986.

[Ti1] J. Tits, Groupes simples et géométries associées, *Proc. Intern. Cong. Math. Stockholm* (1962), 197–221.

[Ti2] J. Tits, Sur les constantes de structure et le théorème d'existence des algèbres de Lie semi-simples, *Publ. Math.* I.H.E.S. **31** (1965), 21–58.

[To] M. L. Tomber, Lie algebras of type F, *Proc. Amer. Math. Soc.* **4** (1953), 759–768.

[Tow1] J. Towber, Two new functors from modules to algebras, *J. Algebra* **47** (1977), 80–104.

[Tow2] J. Towber, Young symmetry, the flag manifold, and representations of GL(n), *J. Algebra* **61** (1979), 414–462.

[Va] V. S. Varadarajan, *Lie Groups, Lie Algebras, and Their Representations*, Springer-Verlag, New York, 1974, 1984.

[vdW] B. L. van der Waerden, Reihenentwicklungen und Überschiebungen in der Invariantentheorie, insbesondere im quarternären Giebiet, *Math. Ann.* **113** (1936), 14–35.

[Vu] T. Vust, Sur la théorie des invariants des groupes classiques, *Ann. Inst. Fourier* **26** (1976), 1–31.

[Wa] Z-X. Wan, *Lie Algebras*, Pergamon Press, New York, 1975.

[We1] H. Weyl, *Classical Groups*, Princeton University Press, Princeton, NJ, 1939; second edition, 1946.

[We2] H. Weyl, Über Algebren, die mit der Komplexgruppe in Zusammenhang stehen, und ihre Darstellungen, *Math. Zeit.* **35** (1932), 300–320.

[Ze] A. V. Zelevinsky, *Representations of Finite Classical Groups*, Springer Lecture Notes 869, Springer-Verlag, Heidelberg 1981.

[Žel] D. P. Želobenko, *Compact Lie Groups and Their Representations*, Translations of Mathematical Monographs, vol. 40, American Mathematical Society, Providence, RI, 1973.

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